A
NEW SYSTEM
OF
ARITHMETICK,
Theorical *and* Practical.

WHEREIN

The SCIENCE of NUMBERS

IS DEMONSTRATED

In a REGULAR COURSE from its FIRST PRINCIPLES,
thro' all the PARTS and BRANCHES thereof;

Either known to the ANCIENTS, or owing to the
Improvements of the MODERNS.

The PRACTICE and APPLICATION to the *Affairs* of *Life*,
and *Commerce* being also Fully Explained:

So as to make the Whole a

COMPLETE SYSTEM of THEORY,

For the Purposes of MEN of SCIENCE;

And of PRACTICE, for MEN of BUSINESS.

By *ALEXANDER MALCOLM*, A. M.
Teacher of the MATHEMATICKS at *Aberdeen.*

T O

THE RIGHT HONOURABLE

William Cruikſhank, *Eſq;*
LORD PROVOST:

James Mooriſon,
John Gordon,
William Mowat,
Hugh Hay,
} *Baillies* :

Alexander Livingſton, *Dean of Guild*;
Alexander Robertſon, *Treaſurer :*

And the Remanent Members of the Town-Council of
A B E R D E E N.

My L O R D, &c.

THE Subject and Deſign of the following Work, with My Relation to the Town of *Aberdeen* as a publick Teacher, naturally directed me to its Governours for Patronage. As the Encouragement I have already met with from the Town, and in particular from its preſent

Magiſtrates

Magistrates and Council, both with respect to this Work, and to my publick Employment, is but the Native Consequence of that Noble Disposition for promoting Learning and all good Education, which is the well-known Character of *Aberdeen*; so, I believe, I should rather offend than please by attempting any Apology for this Address; or running into the Common Way of flattering Dedications. I know how disagreeable it is to Generous Minds, to hear their own just Praises: But I hope you will forgive me, if I avoid the Appearance of Ingratitude, by making this publick Acknowledgement of the Kindness and Civility with which You have used,

My L O R D, &c.

Your Lordship's, &c. *Most*

Obedient Humble Servant,

A L E X^R M A L C O L M.

PREFACE.

WHEN a Subject has gone thro' so many Hands as Arithmetick has done, a new Book cannot want many Prejudices against it: and therefore to send it into the World without some introductory Account of it, is no better than laying it down at random; or, more properly, exposing it. 'Tis an unreasonable Neglect of something, that equally concerns the Author and the World: for if an Author has endeavour'd to do something more useful and complete upon any Subject than has been already done, and thinks he has in some measure succeeded; as the telling the World so, may be done without any Breach of Modesty, so it appears to me equally just and necessary to explain particularly wherein the Improvements and Advantages of the Work lie; that every one may see how far it answers their Purpose, and deserves their Encouragement. It must stand upon its own Basis, no doubt; yet nothing seems more honest and reasonable than this kind of Invitation to look into it. It may be objected, I know, that here is only the Author's Word for this Account, which is a partial Testimony: But if it be consider'd, that he ventures his Credit as well as the Success of his Work, upon a fair Representation, this, I may reasonably hope, will incline the more Candid and Charitable, to believe that it is so. And upon this Hope I presume to give you the following Account of this Work.

ARITHMETICK is a Subject of that Extent, that in some Respects it can never be exhausted; and of that Value, as to deserve all the Study and Pains that can be bestow'd upon it. It is certain, there is no End in the Knowledge of Numbers; but as to a just and rational System of the Science, one would think that can't be a thing still wanted, after so many Books already written on this Subject: Nevertheless, in my Opinion, we are far from having any such thing, in our Language at least; and as to what may be in other Languages, I can only say, That I have not found it in the Books that have come to my hands.

But that I may express my Sentiments upon this Matter a little more particularly, as necessary to introduce an Account of the present Work, I shall first observe, That Arithmetick is to be consider'd in two Respects, viz. either in its *Theory*, which contains the Abstract and Speculative Knowledge of pure Numbers; or in its *Practice*, which contains the Application of that Theory to human Affairs. The Theory is first in order of Science; the Application supposing and depending upon it: So that there can be no Application without some previous abstract Knowledge of Numbers; that abstract Knowledge being the very thing to be apply'd. But then it is to be consider'd, that there is a great Difference betwixt understanding the Sense and Meaning of any Proposition, or of any Rule in Arithmetick, so as to be able to follow its Directions; and knowing the Reason and Demonstration of the Truth of that Proposition or Rule. Hence it is, that there are two very different ways of studying and knowing Arithmetick. The Generality who practise Arithmetick, and even many whose Business requires a Knowledge much above the more common Parts, yet understand little or nothing of the Reason or Demonstration

of

of what they do, becaufe they ftudy not the Theory of Arithmetick, and ask no more than plain Rules for the Practice, fo far as they have ufe for it. But others, confidering Arithmetick as a Science founded upon Principles and Reafon, require a Demonftration for every thing.

Anfwerable to thofe two different Demands, the Books of Arithmetick which we have, are of two kinds; the *Practical* and *Theorical.* The practical Books are moft of them fmall Treatifes of the firft and more fimple Elements and Applications of Arithmetick: But befides that they go a very fhort way into the Science, they have alfo left us without the leaft Reafcn for any thing they deliver, more than what is in fome Cafes evident from the Nature of the thing: Taking all the reft for granted, or leaving the Demonftration to the Theorifts.

The Theorical Writers have treated Arithmetick as a Science, by demonftrating what they deliver: Some of them treat the Subject altogether abftractly, without any particular Application, as *Parfons* in his *Clavis Arithmeticæ*: others with the Theory join alfo the Application, doing more or lefs in it as they have thought fit, as *Ward*, *Tacquet*, and others. Of this Clafs, again, fome begin in the natural Order with the fimple Elements: Others omit thefe, fuppofing them already underftood, and fall in at once into a more advanc'd Theory. Such Elements of Arithmetick we have in *Euclid's* 7th, 8th, and 9th Books; and this Method has been imitated both anciently and of late. *Tacquet* has given us thofe three Books of *Euclid* for the Elements of Arithmetick; placing them before what he calls the practical Arithmetick, which contains the common Principles and Rules, and fome things relating to Progreffions, with the Extraction of the Square and Cube Roots; all very nearly explained and demonftrated, as far as he carries the matter, [excepting one fmall Miftake I have occafionally taken notice of in the following Work; and his Demonftrations of the Square and Cube Roots, which appear to me deficient.] But I could never underftand the Reafon of this Order; he could not certainly mean that thofe Elements of *Euclid* were to be ftudied before the more fimple Elements, which without doubt *Euclid* fuppofed as neceffarily previous to his.

But of all the Works of this Clafs, I have found none which I can reckon a plain, rational and compleat Syftem or Inftitution of the Science of Arithmetick; either from the want of feveral things, even elementary and fundamental in the Science, (which is a common Fault with them all) or being too concife and fhort in other things; or from fome other Difficulty or Fault in the Method; owing, perhaps, to their particular Views and Defigns; but which anfwers not to my Idea of the thing wanted. How unaccountable (for Example) is it in Mathematical Writers, to leave feveral things undemonftrated, to fend us to *Euclid* for others, or give us but very general and imperfect Hints of a Demonftration? But I have done; for to be more particular, would not only be ufelefs, but perhaps be mifconftrued to a worfe Senfe.

FROM this general Account of Arithmetick, and the different Ways of treating it, the Thing wanted will eafily appear to be this, *viz.* A Treatife, wherein the Science is deduced from its firft Principles; and carried on with clear and accurate Demonftration thro' all the fundamental Branches of its Theory and Practice, with the more confiderable Improvements hitherto made in the Science; all difpofed according to the moft eafy and natural Connection and Dependence of the feveral Parts; hereby uniting the whole into one regular and complete Syftem. Again, in fuch a Syftem Numbers muft not only be confider'd abftractly, or purely as Numbers, but we muft alfo confider their Application to particular Subjects, that we may have a compleat Courfe of what we call the practical Arithmetick; which, befides the more fimple Elements of Practice, or fundamental Rules of Operation with pure and abftract Numbers, explains the Application of thofe Rules to the more common and ordinary Subjects of human Affairs.

Such

Such a complete and rational Syſtem of Arithmetick, accommodated to the Purpoſes both of the practical and ſpeculative People, I have endeavour'd to give in the following Work; of the Contents and Order of which, I ſhall give you a more particular Account immediately: But before this, it will be proper to make the following general Reflection upon the Syſtem of Arithmetick, both as to its Theory and Practice; which is this:

ARITHMETICK taken abſtractly, or in its Theory, being the firſt great Branch of the Mathematicks, its Application is to be found not only in the common Affairs of Life and Society, but in all the Sciences that are call'd Mathematical, (which have all their different Uſes in Society.) But then obſerve, that it is not to be expected that a Courſe of Arithmetick ſhould explain ſuch Applications as require the Knowledge of other Sciences; for then we ſhould be obliged to bring into it all the Mathematical Sciences; ſince to underſtand its Application to the Subjects of theſe Sciences, does neceſſarily require our underſtanding the Principles of them. For Example, if 'tis propoſed to find what Part or Parts any leſſer Sphere is of another, the Lengths of their Diameters being known? This is a Queſtion ſolvable by Arithmetick; yet the Reaſon of the Rule goes farther than Arithmetick, for it depends upon Geometry, *viz.* upon that Geometrical Truth, that Spheres are to one another in Proportion as the Cubes of their Diameters; and ſo belongs to Arithmetick only as this is applicable in Geometry, and ſuppoſes the Knowledge of this Science.

From this it is evident, that the Applications proper to be explained in a Syſtem of Arithmetick, are only ſuch as relate to the more ordinary Affairs of Life, which require the previous Knowledge of no other particular Science, and depend immediately and directly upon the Conſideration of the Numbers of things, and ſome other common Circumſtances. Such are all the ſimple Applications of the common fundamental Operations of Arithmetick, either in Whole Numbers or in Fractions; and the Applications of the general Rules of Proportion in the common Subjects of Trade and Commerce: For in all this there is no more requir'd, but a careful Attention to the Senſe of the Queſtion, and the true Effect of the Rules of Arithmetick.

Again it is to be obſerv'd, That as the Theory of Arithmetick is an abſtract Science, independent of all thoſe Subjects to which it may be apply'd, it is therefore neceſſary that we have a complete Syſtem of the Theory of Numbers, conſider'd purely and abſtractly by themſelves; this being preſuppoſed in the Solution of all Queſtions in other Sciences, which have any Dependence upon Numbers. The next thing I obſerve here, is, That tho' there be many Truths diſcovered in the Theory of Arithmetick, of which there has been no Uſe or Application yet found, this is no reaſon why thoſe things ſhould be neglected or kept out of the Syſtem; they are ſtill a Part of the Science, which we ought to enlarge more and more, as far as we can: One Age may find the Uſe of the Theory which a former has invented; as undoubtedly has been the Caſe, with reſpect to moſt part of the Theory both of Arithmetick and Geometry. I ſhall but add this one thing more, *viz.* That tho' many things in the Science of Numbers were ſuppos'd to be of no particular Uſe in human Affairs, yet as the Mind of Man is made for Knowledge and Contemplation, and the Pleaſure ariſing from the Perception of Beauty and Order in other things, is allow'd to be worthy of rational Natures; the Contemplation of the ſurprizing Connections, the beautiful Order and Harmony of Relations and Dependencies found among Numbers, is not leſs reaſonable: And if to this be join'd the vaſt Extent of the Uſe and Application of Arithmetick, the Reaſonableneſs and Neceſſity of explaining the Theory of Numbers ſo largely as I have done, will eaſily be allow'd.

I ſhould now come to the Contents of the following Work, but as ſome particular Circumſtances oblige me to take notice of two late and well-known Authors, Mr. *Hill* and Mr. *Hatton*, I ſhall firſt diſcuſs what I think neceſſary to ſay as to their Works.

Mr. *Hill's* Book, which he calls *Arithmetick in Theory and Practice*, is remarkable chiefly for the very uncommon Recommendation it carries with it from a very conſiderable Maſter. We are told by Mr. *Ditton*, That take this Author purely as an Arithmetician,

metician, he has not only done more and much better than *Wingate, Cocker, Leyburn,* or any other of the Writers in our Tongue, but indeed all that can be done by Arithmetick; and therefore (says he) if no other Book on this Subject comes out till this Performance is really mended, I'm satisfy'd we shall have no new Book of Arithmetick very soon.

Now here was such a Defiance, and from such a Hand, that they were bold enough who ventur'd first to write after it, and even without the least Apology, or Notice taken of this Challenge, as several have done: Whatever Reason others thought they had for such a Conduct, I thought it necessary for my own Vindication to make the following Remarks on this Book and its Recommendation.

That Mr. *Hill* has several things that are not common, I do acknowledge; but for his having done much better than all that went before him, 'tis not my Business to determine; what I'm concern'd in chiefly, is the Assertion of his having done all that can be done by Arithmetick, a thing I was much surpriz'd to hear from so good a Judge as Mr. *Ditton*; and because if this be true, then what I offer to the World must be either impertinent or superfluous; it can't be thought out of my Road if I enquire a little into the Truth of this Assertion. By the mention Mr. *Ditton* makes of *Algebra,* it appears to me, that he would have nothing admitted into Arithmetick that is any way owing to *Algebra.* Now, supposing this were reasonable, yet the Book in question will still be found both very defective in what it ought to contain to answer so great a Character, and also to have many things that belong not to a pure Treatise of Arithmetick. In the first place, with what Truth and Justice can it be said, That a Book contains all that belongs to Arithmetick, and that one needs learn no more, (as he also says) which, (besides that there is no Demonstration, and consequently no Science) wants many things that are fundamental and necessary, and yet do not absolutely depend upon *Algebra*; (tho' they may be made easier in many things by its help) particularly in the Doctrine of Proportion: For tho' we have here several Propositions relating to this Subject, yet we are very far from having any thing like a just and orderly Treatise of Proportion: Nor (to mention no more of its Defects) have we any of the rest of the fundamental and curious Theory of Numbers, contain'd in *Euclid's* 7th, 8th, and 9th Books. Again, if we must exclude what is any way owing to *Algebra,* then most of what is uncommon in Mr. *Hill,* as upon *Progreffions, Interest, Logarithms, Combinations,* and *Extraction of Roots,* do not belong to Arithmetick: And if these belong to *Arithmetick,* notwithstanding their Dependence upon *Algebra,* then so must a great many other things not to be found in Mr. *Hill's* Book. But I have said enough, and shall leave you to judge by the following Work, whether that Book contains all that can be done by *Arithmetick,* and consequently what to think of this extraordinary Recommendation, which indeed is more faulty than the Book itself.

Mr. *Hatton's* Book, which I have here in my view, is his *Intire Syftem of Arithmetick*: If this Book had answer'd the promising Title, my Labour had been prevented; but I could not help judging otherwise of a Book that not only leaves us without Demonstration in most things, and sometimes gives us a mere Proof of a particular Example, instead of a general Demonstration; but which, in a word, comes very far short both of the Contents and Order due to an *Intire Syftem.* As I'm no further concern'd in the Criticism of another Man's Work, than it is necessary to vindicate my own, I shall be content with this general Reflection upon this Work, and leave it to an impartial Comparison to justify what I have alledg'd, and determine whether there was not yet wanting a *more Intire Syftem.* There is one thing more I must say here, viz. That as I think it is every one's business who writes upon any Subject, to discover the Errors (especially if they are of any consequence) committed by others, so I hope there will be no Misconstruction made of my Design, in exposing some Errors I have found in this or in any other Author: It is the Treatment I expect my self, and shall receive without complaining.

plaining *. But I'm oblig'd for the fake of the Publick, to obferve here, that as Mr. *Hatton* has found fault with Dr. *Harris's* Rule for the difcompting of *Simple Intereft*, for Money paid before it is due; fo he ought in juftice to have told the World, That the Tables of Difcompt in his own *Index to Intereft*, printed *Anno* 1711, are calculated by the fame falfe Rule; that no body may be longer impos'd upon by them. And that you may not think the Confequence inconfiderable, take this Example: The Difcompt of 1000 *l.* paid 90 Days before due, is by his Tables *l.* 13.9571, the Difcompt being at 6 *per Cent.* whereas, according to the True Rule, the Difcompt ought to be *l.* 14.57882

I proceed now to a more particular Account of the following Work, which is divided into fix Books.

BOOK I.

In this Book I have largely explain'd and demonftrated the firft fimple Principles and fundamental Operations of Arithmetick in *Integers* or *Whole Numbers:* In which, after the Principles and Rules for the Management of pure and abftract Numbers, I have feparately explain'd the Ufe and Application of thefe Rules to particular Subjects, fuch as occur in human Affairs.

BOOK II.

Here you have fully handled the Doctrine of *Fractions*, where I have firft explain'd (in a way I think very eafy and demonftrative) the general Nature and Theory of Fractions, as a neceffary Foundation for underftanding the Reafon of the Practice; which I have next fully explain'd both in *Vulgar* and *Decimal Fractions*, as they are diftinguifh'd. Only what we call *Infinite* or *Circulating Decimals*, are referr'd to *Book* 5. for the fake of the Demonftration.

Obferve, As thefe two Books contain the firft and fundamental Principles and Rules of Arithmetick; and as the right underftanding of the Foundations of any Science is of great Importance, I have therefore enlarg'd and improv'd every Part almoft with fuch particular Explications and Rules, as will, I hope, be of great ufe for attaining to a juft and perfect Idea of this Science in its Fundamentals, and a more mafterly Practice.

BOOK III.

Contains the Doctrine of the *Powers* and *Roots* of Numbers; wherein I have firft particularly explain'd, the Nature and Theory of thofe Numbers call'd *Powers* and *Roots*. After this you have the Rules for raifing or forming *Powers*, and *Extracting Roots* in Integers and Fractions, where I have explain'd Sir *Ifaac Newton's* famous Rule call'd the *Binomial Theorem*, and fome other curious things relating to the *Extraction* of *Roots*. You have here alfo what is called the Arithmetick of *Surds*, which contains a more particular Application of the preceding Theory to *Roots*, efpecially thofe called *Surds*. Laftly, you have all the Propofitions of the 2d Book of *Euclid*, which are applicable to *Numbers*, with fome others of the fame kind.

Obferve, As to the Contents of this Book, that excepting the common Rules for extracting the *Square* and *Cube Roots*, all the reft of this curious Branch of Arithmetick is to be found only in our Books of *Algebra:* becaufe the Ufe of it is above the common Affairs of Life, and occurs chiefly in the higher Applications of the *Algebraick* Art; and alfo becaufe the Demonftration of it can hardly be made without the help of *Algebra*. But as it is directly and immediately a Part of the Theory of Numbers; which does indeed no otherwife belong to a Treatife of *Algebra*, than any other thing in Arithmetick,

* *The Particulars I have cenfured in Mr.* Hatton's *Book, you'll find in* Book 6. Chap. 6. Queft. 6. *and in* Chap. 11.

a

which

which may be demonftrated by the help of the *Algebraick* Method. Alfo, fince I have taken that Method of Demonftration (of which I fhall give you a particular Account afterwards) I have therefore given it its due place in the *Syftem of Arithmetick.* I muft alfo obferve, that tho' the Writers of *Algebra* have taken this Part of the Theory of *Arithmetick* into their Province, yet it is not, in our Language, treated fo fully and particularly as it ought to be; many things being left without Demonftration that to me feem far from being felf-evident.

For the Extraction of *Roots,* efpecially thofe above the *Square* and *Cube,* there are eafier Methods, owing alfo to the *Algebraick* Art; but as they exceed the Limits prefcrib'd to this *Syftem,* they muft therefore be fought elfewhere.

B O O K IV.

Contains the *Doctrine of Proportion* in all its Branches, as diftinguifhed into *Arithmetical,* *Geometrical* and *Harmonical.* In each of which, as I have endeavour'd to make the fundamental things clear and plain, fo I have omitted nothing worth knowing, in this great and ufeful part of Arithmetick, that I could any where find, or that my own Study could furnifh: Whereby, as you have all that our common Books contain, fo you have many other things to be found only in fuch Authors as are not in every body's hands; and many things intirely new, for what I know. And, in both thofe laft two kinds, befides what is mix'd here and there, there are fome more confiderable Additions; particularly upon *Arithmetical Progreffions,* in *Chap.* 2. §. 2. All that is from *Schol.* 2. (after *Probl.* 2.) is intirely new. The *Chapters* 5, and 6. with the *Appendix* to this Book, contain things uncommon, and for the moft part altogether new; (fee *Contents* more particularly.) So that I dare prefume to fay, you have here a more compleat Syftem of the Doctrine of Proportions than can be found elfe where, in our Language at leaft.

As to the Subject of *Chap.* 6. which is *Harmonical Proportion,* I have this Obfervation to make, That as Mufick in its firft Principles depends altogether upon Numbers, fo the Knowledge of the Application of Numbers to Mufick, which I may call the Arithmetical Theory of it, is fo very ufeful and entertaining, that 'tis pity it were fo little underftood, as I doubt it is, both by the Practifers and Lovers of Mufick. What was proper or neceffary to be done in this Work, concerning that Application, I have done it; and if any one wants a particular Treatife upon this Subject, they will find it in a Book call'd, *A Treatife of Mufick, Speculative, Practical, and Hiftorical;* which is to be found with the Bookfellers to whom the prefent Work belongs.

B O O K V.

This Book is a Mifcellany of various things, which are not comprehended under one common Name; and confifts of VI. Parts, in as many different Chapters; whofe Contents are as follow.

1. The Doctrine of *Prime* and *Compofite* Numbers; a fundamental and curious Branch of the Theory of Arithmetick.

This is a great Part of the Doctrine of *Euclid's* 7th, 8th, and 9th Books of *Elements;* which contain, befides, many things relating to the Doctrine of *Proportion;* but thofe I have put in their due Place with the reft of that Doctrine, which is not fo complete in *Euclid* as it has been made fince: but as my Method of Demonftration is generally different from his, (tho' in fome things there can't be a better than his, and perhaps no other;) fo I have not only deliver'd this Theory in a different, and, as I think, a more natural Order; but by means of the *Algebraic* Method, I have gain'd thefe Advantages, *viz.* That feveral things are made *Corollaries* to others, which have a fufficiently tedious Demonftration in *Euclid.* Again, feveral Propofitions are made univerfal, which are limited

mited in *Euclid* to a few particular Cafes: And in others, which can be prov'd only by an Induction of Particulars, I have made the Univerfality of the Induction more clear and evident, by another Method of Reafoning.

There are here alfo many things which are not in *Euclid*; Part of which I met with in fome rare Books, and others occurr'd to my own Study and Obfervation; particularly the 3ᵈ Section is intirely new. I fhall mention but one thing more, that is, a *New* and very Eafy Way of finding all the *Prime* and *Compofite* Numbers within any given Limit: of which I have given an Example, extended only to 999. The Form of the Table in which they are collected, is much the fame with that in Dr. *Pell's* Edition of *Brancker's Algebra*; tho' the Rule by which I have compofed it be vaftly more eafy than what's given there.

2. The curious Theory of *Figurate Numbers*; a thing but juft touch'd upon in any *English* Book, of my Acquaintance. I have met with more of it in fome others, but either without Demonftration, or fo much out of my Method, that I could make no ufe of it. And here the Advantage of the *Algebraick* Method was manifeft, by which, feveral of thofe things are very fimply and eafily demonftrated, that otherwife have a very difficult and tedious Demonftration: and without which other things could not, I doubt, be demonftrated at all. To the fame means alfo I owe feveral things here, that I found in none of my Authors; whereby I have carried this Part further, and, by putting the whole together in a juft Order, have given it a more perfect Form than I have any where found it in.

Here you have a new *Canon* for the *Coefficients* of the **Powers of a Binomial** Root; and feveral curious Propofitions, relating particularly to Square Numbers: With Rules for fumming the Series of the Squares and Cubes of the *natural Progreffion* of Numbers 1. 2. 3. 4. *&c.* without actually raifing thefe Powers and adding them together; and a Method of raifing *Canons* for fumming any of the higher Powers.

As to the Ufe of this Part, whatever elfe it may be (which in *Mathematical* Affairs is confiderable) you have thro' the whole, remarkable Examples of what I formerly mention'd, *viz.* Of beautiful and furprizing Order and Connection among Numbers.

3. Of *Infinite Series* of Numbers; particularly of decreafing Geometrical *Progreffions*, (fome ufeful Applications of which you have in the next Chapter) and of thofe *Encreafing* Series, which are the chief and fundamental things of what the Mathematicians call the *Arithmetick* of *Infinites*; of which they have made a noble Ufe in *Geometry*; having hereby particularly found many ufeful practical Rules, for the Menfuration of *Solids* and gauging of Veffels. What I have done here belongs properly to *Arithmetick*. The Application of it to *Geometry* you'll find in *Sturmy's Mathefis Enucleata*, or *Ward's Introduction*. But the whole Doctrine and Application at large, is to be fought from the celebrated Author of it, Dr. *Wallis*.

4. The Theory and Practice of *Infinite* or *Circulating Decimals* (referr'd to this Place for the fake of the Demonftration) which, with what is already done in *Book* 2ᵈ, *Chap.* 2ᵈ, makes a compleat Syftem of *Decimals*.

Dr. *Wallis* is probably the firft, as he has himfelf obferv'd, who has diftinctly confider'd this curious Subject of Circulating Decimals. He has given us the fundamental Theory of it, but without Demonftration; nor has he meddled with the practical Part, or Way of managing Infinite *Decimals* in Arithmetical Operations. Mr. *Brown*, in his Decimal Arithmetick, has handled but one fingle Cafe of the Practice, and that not completely neither. Mr. *Cunn.* (who is the laft Author I know upon this Subject) in his *Treatife of Fractions*, has in his way given us all that Dr. *Wallis* fays upon the Theory, yet without any Demonftration, and a few other obvious things, tending more immediately to the Practical Part; which he has handled at full length, giving us Rules for all Operations and all Cafes: But as he demonftrates none of thofe Rules, (which perhaps he referv'd for another Work) he has alfo chofen to exprefs them in fuch a manner, as to fet the

Reafon as far out of view as poſſible, which has this Effect, that in the Rules of Multiplication and Diviſion (which are the more complex and difficult Parts) his Directions are not ſo eaſily follow'd ; and are beſides much harder for the Memory than the Method I have choſen, which depends all upon the eaſy and natural Explication of one ſingle Propoſition, *viz.* The finding the finite Value of (or Vulgar Fraction equal to) any circulating Decimal: for tho' the Demonſtrations are omitted, the Rule ought to be as ſimple and eaſy as poſſible. But I muſt obſerve this further Effect of Mr. *Cunn*'s Way of delivering theſe Rules, That by themſelves one could never, or very hardly, be led into the Reaſon of them, nor conſequently into the way I have choſen; ſo that it will be the more eaſily believ'd that the Rules I have given, are the Effect of Speculations made upon this Subject, before I ſaw this Book; which I mention for this Reaſon only, that I may not be thought ungrateful to one whom I acknowledge the firſt Author upon this Practice, from whom therefore I might otherwiſe be ſuppoſed to have borrowed or deduced all that I ſay; and yet I do acknowledge I owe him one or two uſeful Hints. I have only one thing more to add, *viz.* That his Rule for the Addition of *Circulates* having compound *Repetends* is inſufficient for a general Rule; it will bring out the true Anſwer in ſome Caſes, but is not univerſally good for all Caſes: the comparing it with the Rule I have here demonſtrated will ſhew the Difference, and the Truth of what I ſay.

5. The *Logarithmick Arithmetick*; wherein the Nature, Conſtruction, and Uſe of thoſe admirable Numbers call'd L O G A R I T H M S are explain'd and demonſtrated.

The Manner of conſtructing or making Logarithms, which I have explained here, is that of the Noble Inventor, the Lord *Neper*, becauſe its Demonſtration is more ſimple and eaſy, tho' the work itſelf vaſtly more tedious than other Methods which have been diſcover'd ſince, by means of a deeper Application of the *Algebraick Art* than my limits allowed me to uſe here. My Purpoſe is however ſufficiently anſwer'd; for as every one who would underſtand the Reaſon and Uſe of *Logarithms*, is not under any neceſſity of conſtructing them, that being often done already; ſo I deſign'd chiefly what I think is moſt generally demanded, that is, (1.) To demonſtrate the *Origin* and *Nature* of thoſe Numbers, or ſhew that there are really ſuch Numbers to be found, as we define *Logarithms*; which could not be better or more naturally done than by the Method of the Inventor. And then, (2.) to explain and demonſtrate their Uſe and Application; which is the ſame, whatever way they are calculated or conſtructed.

I ſhall ſay but this one thing more, *viz.* That as thoſe other Methods of Conſtruction are chiefly owing to Sir *Iſaac Newton*'s Binomial *Theorem*; ſo far as I have explained that *Theorem*, (which is only ſo far as relates to *ſimple* or *proper Powers*, i. e. having Integral Indexes) I have ſo far alſo made their way eaſy, who would ſtudy thoſe other Rules of Conſtruction wherein that Theorem is alſo apply'd to *Roots*; which Rules they will find no where more eaſily and fully explain'd than in *Ronayne*'s Algebra.

6. Of the *Combinations* of Numbers, a Part of Arithmetick which has been but very little and generally handled by our *Engliſh* Writers; and as little by others that have fallen in my way. We have indeed moſt of the fundamental Propoſitions of it, in *Hill*'s Arithmetick, yet far ſhort of the Length I have carried it to here. As the thing is in itſelf curious, and not without conſiderable Uſe, eſpecially in the Calculations of Chances, I have explain'd it the more particularly. Here alſo you have another Demonſtration of the *Binomial* Theorem for *Coefficients*.

B O O K VI.

Contains the Application of the Doctrine of *Proportion* to the Common Subjects of Human Affairs: Wherein I have gone thro' a large and complete Courſe of all the Common Rules and Branches of this Application. I have labour'd to make the Rules as plain and intelligible as poſſible; and at the ſame time expreſs them ſo, as the vaſt

Extent of their Ufe may eafily appear; and young People may not be fo limited in their Notions of thofe things, as not to be able to go further than the few Examples to which they have found them apply'd in Books, or by their Teachers; or fuch Examples as are ftrictly of the fame kind, and propofed in the fame manner with thefe: as I have often found to be the Cafe in the Courfe of my Bufinefs and Experience in thofe Matters. The beft Remedy of which, is to underftand the Reafon and Demonftration of every Rule, and fee the Applications of it in a great Variety of Subjects and Circumftances. Therefore I have fhewn the Reafons of all the Rules by their Dependence upon the preceding Theory. But left any thro' neglect, or fome other fault, fhould not underftand that Theory, I have here given fome other Demonftration of the chief and moft ufeful of thofe Rules. And to make the Application complete, I have given you not only a fufficient Number of Queftions of common Ufe and Occurrence in every Branch; but alfo a great many that are uncommon and curious, the ftudying of which will ferve this very ufeful Purpofe, *viz.* to lead one to a clearer and readier Apprehenfion of the Application of the Rules of Arithmetick, and efpecially of Proportion, which is the moft important and difficult thing in the Practical Arithmetick.

The Applications relating to the *Intereft* of Money being of great Concernment to Society, I have explained and demonftrated thofe at large. And here I found myfelf neceffarily engaged in the Examination of a Queftion wherein Sir *Samuel Moreland* and Mr. *John Kerfey* have widely differ'd. The Queftion is about the Calculation of the prefent Worth of an *Annuity* to continue any number of Years, difcompting fimple Intereft. Whether Sir *Samuel's* Book, which he calls *The Doctrine of Intereft,* wherein he finds fault with Mr. *Kerfey's* Rule, which is in his *Appendix* to *Wingate's* Arithmetick, was written before *Kerfey's* Death, or whether he ever faw it, or gave it any Anfwer, is what I know not; but the Difference feem'd to me too confiderable to pafs over. Upon the moft careful Examination, I was determin'd to Mr. *Kerfey's* fide: tho' I was very foon afterwards furpriz'd to find my Opinion contradicted by the Mathematical Writers, who have taken the other fide, and form'd their Rules upon *Moreland's* Foundation; as particularly by *Ward,* with this Remark, that " *Moreland* has detected feveral material " Errors committed by *Kerfey* upon *Wingate.*" This put me upon a more narrow Examination of the Queftion, which ended in a further Confirmation of my former Opinion; and yet what Mr. *Ward* and others have done merely as Mathematicians, is right; they have affum'd a certain ftate of the Queftion, and upon that raifed Rules which are good upon the Juftice of that State of the Queftion, but not otherwife: and therefore in the Defence of Mr. *Kerfey's* Rule, I differ from *Ward,* and others who have taken that Method, not as to any pure Mathematical Truth in Numbers, but merely as to a point of Right and Equity, in the manner of ftating a Queftion betwixt Man and Man, according to the Conditions previoufly agreed to: But that every body may judge and chufe for themfelves, I have given the Rules and Reafonings upon both fides.

As to the Rule of *Pofition,* or *Falfhood,* which is common enough in Books of Arithmetick, I have omitted it, becaufe it is of little or no Ufe in real Bufinefs, and very limited in its Application: Befides, whoever has the leaft Smattering of the Algebraick Method of folving Queftions, can do all, and much more, than this Rule teaches.

Of the Method of Demonstration ufed in the following Work.

I have every where endeavour'd to take the moft eafy and natural way the thing would admit of. In the firft and fimple Parts there is but one way almoft to be taken; but in the more complex and difficult Parts, as there is room for a Choice, I have generally ufed the *Algebraick* Method, as what is natural and proper to *Arithmetick,* and the moft eafy and plain Method that can be ufed in moft Parts of this Science, and without which many ufeful and curious things could not be demonftrated. I have not fuppofed the Student of Arithmetick already acquainted with Algebra; but have gradually explain'd

the Principles and Rules of it, as far as my Purpose requir'd. As *Algebra* is nothing else but an univerfal Method of reprefenting Numbers, and reafoning about them, fo it very naturally belongs to Arithmetick: And in the Opinion of the Great Sir *Ifaac Newton*, who calls it the *Univerfal Arithmetick*, makes, with what in diftinction from it he calls the *Vulgar Arithmetick*, but one complete Art of Computation. But my Defign not reaching to a complete Syftem of Arithmetick in this larger Senfe, I have done no more as to *Algebra* than is neceffary for demonftrating the Syftem of Arithmetick in the more ftrict Senfe.

I have indeed been ask'd, why any thing is brought into a Treatife of Arithmetick, which ftands in need of the *Algebraick* Art, or can be better done by that means than otherwife, and not rather referr'd to a Treatife of *Algebra?* The Anfwer was obvious, *viz.* That wherever thefe things are placed, they belong to the Syftem of Arithmetick: And for the *Algebra* requir'd to the Demonftration, if one has already learnt it in a more exprefs and particular Study of that Art, it is well; but if not, 'tis juft as proper and eafy to learn it in a Courfe of Arithmetick, as naturally belonging to this Science. And if it is again ask'd, Why then I have not extended this Work to all the Parts of the *Algebraick* Art, and thereby made a Syftem of Arithmetick more complete? I anfwer, That having the Choice of my Subject, I have given it fuch limits as I thought convenient, and done fuch a Work as I thought was moft wanted: Thofe who incline to make a more particular Study of the *Algebraick* Art, muft feek it elfewhere. But if what is done here, both as to the Principles and Application of *Algebra*, be well underftood, it will, I believe, prove an ufeful Introduction to the higher Parts of this admirable Art, and a powerful Incitement to the further Study of it; when it is confider'd, how the moft fimple Elements of it are fufficient for acquiring fuch a Knowledge of Arithmetick as can't be obtain'd without it in many things, and in others not without much greater difficulty.

I hope then there will no Difcouragement arife from a Profpect of Difficulty in this Method, by fuch as are willing to ftudy Arithmetick in a reafonable manner: For tho' there are difficult and abftrufe things in the *Algebraick* Art, yet all the Principles and Rules of it ufed in this Work, are in effect no more than a particular kind of Language; or rather a compendious way of reprefenting and comparing Numbers and the Effects of their Operations: which may be learnt with a little pains, in two Leffons, or three at moft; and as they are explain'd and apply'd by degrees, it will become eafy and familiar as you proceed.

For thofe who would ftudy only the Practical Part, without the Theory and Reafons of Things; they will find what they want in the firft, fecond, and fixth Books, with the fecond Chapter of Book III. In all which, let them pafs over the *Demonftrations.* And if they would go further, they may read the *Problems* in the fourth and fifth Books.

It remains that I explain the Meaning of a few Names ufed in the following Work, for different kinds of Propofitions.

A DEFINITION is the Explication of the Ufe and Meaning of any Word or Term of Art; as of this itfelf and the following.

An AXIOM is a Propofition whofe Truth is felf-evident.

A THEOREM is a Propofition whofe Truth is to be demonftrated.

A LEMMA is a Propofition to be demonftrated; and which is premifed to fome other, to ferve as a Principle for the more eafy Demonftration of this other.

A PROBLEM is a Propofition of fomething to be done or difcover'd.

A COROLLARY is a Propofition gain'd in confequence of another, whofe Truth is evident from the Truth or Demonftration of that other.

A SCHOLIUM is fome further Explication relating to what precedes.

Obferve, In the *Demonftrations* of the following Work, when any former Propofition is quoted, it's underftood to be in the fame Book and Chapter in which it is quoted, unlefs it is otherwife expreffed.

A

A
SHORT HISTORY
OF
ARITHMETICK.

THAT Arithmetick was very early in the World, no body can doubt, becaufe the Idea of Number arifes from all things about us. In the beginning, while the Way of Living was fimple, and things were in a manner common, the Knowledge of Numbers would make a fmall Progrefs: But when *Property* and *Commerce* began to be eftablifhed, Men would foon find the Neceffity of enquiring into the Nature of Numbers, and contriving an *Art* of *Numbering*; without which no Bufinefs can be carried on. This was, no doubt, very rude at the firft, and improved by degrees; as all our Knowledge is: But where, and by whom, Arithmetick received its firft Form of an *Art* or *Science*, we know little about it. If the *Phœnicians* were, as it is conjectured, the firft Merchants after the Flood, (and before that we know nothing of the Affairs of Mankind) then it is probable, the *Art* began among them; by whom *Trade* and *Arithmetick* were carried into *Egypt*; and here, 'tis thought, began the *myftical* Application of Numbers: For the *Egyptians* explained every thing by thefe; the Nature of the *Gods*, of *Human Souls*, the *Virtues*; in fhort, for every thing *divine* and *human*, they found fome Symbol or Reprefentation in Numbers: Hence we hear of the wonderful Virtues and Properties of fome particular Numbers, as *One*, *Two*, *Three*, *Four*, *Seven*, and *Nine*. From *Egypt* this Knowledge paffed into *Greece*, which added its own Improvements to the myfterious Part; of which a great deal is to be feen in *Plato*; the Life of *Pythagoras* by *Jamblichus*; and more lately in the Commentators upon *Boethius's* Arithmetick. Now we are come to the Country where we may expect to find the firft diftinct Rudiments of the Science.

The firft thing Men were obliged to do to make their *Ideas* and Knowledge of Numbers ufeful in Society, was to eftablifh fome Method of *Notation*, and then upon this found an Art of *Computation*: after this they would gradually enquire into the *Relations* and *Properties* of Numbers; and fo the Science went on.

The *Greeks*, *Hebrews*, and other Eaftern Nations, ufed a Notation by the Letters of their Alphabet. The *Greeks*, particularly, had two different Methods; the firft was much the fame with the *Roman* Notation, explain'd in *Chap.* 2. *Book* 1. of the following Work, which is a very imperfect Method. Afterwards they had a better Method, in which the firft nine Letters of their Alphabet reprefented the firft Numbers from One to Nine, and the next nine Letters reprefented any Number of Tens from One to Nine, that is, 10, 20, 30, &c. to 90. Any Number of Hundreds they expreffed by other Letters,

<div align="right">fupplying</div>

supplying what they wanted with some other Marks: And in this Order they went on, using the same Letters again with some different Marks to express *Thousands, Tens of Thousands, Hundreds of Thousands,* &c. As to this Method, 'tis to be observ'd, that they were upon the very Point of discovering the *Arabian* Notation: For, as they made the Progression to 9, they wanted but one Step further, *viz.* Instead of using other 9 Letters, to make the same 9 change their Values in a decuple Progression according to their Places, which would in course discover the Necessity of a Character that of itself signifies Nothing, only fills up a Place.

The Manner of their *Computation,* (*i. e.* of *Addition, Subtraction,* &c.) and the Difficulty of it, especially in great Numbers, we may easily discover from the *Notation.* As to any express Treatises upon the *Art* of *Computation,* they have left us none. There is a Commentary by *Eutocius,* upon *Archimedes's* Treatise of the Dimensions of a Circle ; and some Fragments of *Pappus,* in Dr. *Wallis's* Works, which relate particularly to the Work of *Multiplication,* and shew us the great Difficulty of their Practice, owing to the imperfect Notation.

The most perfect Method of *Notation,* which we now use, was owing to the Genius of the *Eastern* Nations; the *Indians* being reckoned the Inventors of our Notation; which we call the *Arabian,* because we had it from them, and they from the *Indians,* as themselves acknowledge. When the *Indians* invented this Method, and how long it was before the *Arabs* got it, we cannot tell: These things only we know, 1. That we have no ground to believe, the antient *Greeks* or *Romans* knew any thing about it: For *Maximus Planudes,* the first *Greek* Writer who treats of Arithmetick according to this Notation, lived about the Year of Christ 1370, as *Vossius* says; or about 1270, according to *Kircher* ; long after the *Arabian* Notation was known in *Europe :* And owns for his Opinion, that the *Indians* were the Inventors, from whom the *Arabs* got it, and the *Europeans* from them. 2. That the *Moors* brought it into *Spain* ; whither many learned Men from other Parts of *Europe* went to seek that, and the rest of the *Arabick* Learning (and even the *Greek* Learning, from *Arabick* Versions ; before they got the Originals themselves) imported there by the *Saracens.* As to the Time when this new Art of Computation was first known in *Europe, Vossius* thinks it was not before the Year 1250 ; but Dr. *Wallis* has, by many good Authorities, proved that it was before the Year 1000 ; particularly that a Monk called *Gerbertus,* afterwards Pope by the Name of *Sylvester* II. who died in the Year 1003, was acquainted with this Art, and brought it from *Spain* into *France,* long before his Death. The Doctor shews also, that it was known in *Britain* before the Year 1150, and brought a considerable length, even in common Use, before 1250, as appears by the Treatise of Arithmetick of *Joannes de Sacro Bosco,* who died about 1256.

Tho' the numeral Figures which we now have are a little different from what the *Arabians* use, having been changed since they came first among us ; yet the *Art* of *Computation* by them is still the same.

Having said all that's necessary about the *Notation* of *Numbers,* we shall go back again, and see what kind of *Science* of *Arithmetick* is to be found among the Antients, with the Progress of it till now.

The oldest Treatise extant upon the *Theory* of *Arithmetick,* is *Euclid's* 7th, 8th, and 9th Books of Elements; wherein he gives us the Doctrine of *Proportion,* and that of *Prime* and *Composite* Numbers. Both of which have received Improvements since his time, especially the former. The next, of whom we know any thing, is *Nicomachus* the *Pythagorean,* who wrote a Treatise of the Theory of Arithmetick, which consisted chiefly of the Distinctions and Divisions of Numbers into certain Kinds and Classes, as *Plain* and *Solid, Triangular, Quadrangular,* and the rest of the Species of *Figurate* Numbers (as they called them) Numbers *Odd* and *Even, &c.* with some of the more general Properties of
the

the several kinds. As to the time in which *Nicomachus* lived, some place him before *Euclid*; others long after. His Arithmetick was published at *Paris* 1538. What kind of Work it is, we may guess by the *Latin* Treatise of Arithmetick of *Boethius* the Philosopher, who lived at *Rome* in the time of *Theodorick* the *Goth*; and is the next remarkable Writer extant upon this Subject. He is supposed to have seen and copied most of his Work from *Nicomachus*.

From this Work of *Boethius*, with a few small Abstracts of the same nature, made very long after his Time, as that of *Psellus*, and *Jodochus Willichius*, both in *Greek*; some have said that the antient Arithmetick consisted of nothing else but these Divisions and Distinctions of Numbers. I confess I was surprized to find this Account from such an Author as *Wolfius*, to whom *Euclid* is no Stranger; whose Books contain things much more important in the Science of Arithmetick than these Distinctions; and want many of them, that are in *Boethius* : For *Euclid* speaks nothing of the *Figurate Numbers*, and their various Species and Classes; except what relates to Squares and Cubes. And, on the other hand, *Boethius* has very little of *Euclid's* Doctrine.

We must come next to the Times when the *Arabian* Notation was known in *Europe*; after which we find many Writers both upon the *Theory* and *Practice*. The oldest of them, who is very considerable, is *Jordanus* of *Namur*, who flourish'd about 1200. His *Arithmetick* (from which I have taken several things) was published and demonstrated by *Joannes Faber Stapulensis* in the fifteenth Century, (who has given us himself a Compendium of *Boethius*) soon after the Invention of Printing. It's altogether upon the Theory; and contains most of what *Euclid* and *Boethius* have, and many other curious Theorems. The same Author wrote also upon the new Art of Computation by the *Arabick* Figures, and called this Book *Algorismus Demonstratus*; the Manuscript of which, Dr. *Wallis* says, is in the *Savilian* Library at *Oxford*. But it has never been printed, as I know.

As Learning advanced in *Europe*, so did the Knowledge of Numbers; which by degrees received large Improvements both in the *Theory* and *Practice*, owing in a great measure to a more perfect Notation. To trace out every Step in that Improvement, is impossible; therefore I shall only name a few of the remarkable Writers after *Jordanus* and *Sacro-Bosco*, both named already. As to the Writers, these were most remarkable in *Italy*, viz. *Lucas de Burgo*, about the Year 1499, whose Arithmetick, which is both Theorical and Practical, Dr. *Wallis* commends much : *Nicholas Tartaglia*, whose Work is chiefly *Practical*. He is called by some the *Prince* of the *Practitioners*; which must be understood only for his own Times. In *France*, there were *Clavius* and *Ramus*. In *Germany*, *Stifelius* and *Henifchius*. In *England*, *Buckley*, *Diggs*, and *Record*. All these, and many more, were before the Year 1600. But since that, our Writers are almost innumerable.

As to the Improvements made since the *Arabick* Notation was known in *Europe*; besides many things in the Theory, particularly in the Nature of *Progreffion*, both Arithmetical and Geometrical, in the Nature of *Powers*, and in the *Extraction of Roots* and the *Combinations of Numbers*, which we do not so well know the History of; there are a few very considerable Improvements, in the *practical* Part, of which we can give a better Account. But that I may connect the Antient and Modern History, we must go back to the second Century of *Christianity*, in which *Claudius Ptolomeus* lived, who is supposed to be the Inventor of the *Sexagesimal Arithmetick*; which was a new Method of *Notation*, and consequently of Computation, designed to remedy the Difficulty of the common Method, especially with regard to *Fractions*. The Nature of it was this : Every Unit was supposed to be divided into 60 Parts, and each of these Parts into 60 Parts, and so on; hence any Number of such Parts were called *Sexagesimal* Fractions. And to make the Computation in Integers also more easy, he made the Progreffion in these also *Sexagesimal*. Thus, From one to fifty-nine were marked in the common way; then sixty was called a *Sexagena prima*, (or first *Sexagesimal* Integer) and marked with the Sign of *Unity*

b · and

and one single Dash over; so sixty was thus expressed I'. Two sixtys, or 120, thus II'; and so on to 59 times 60, (or 3540) which is LIX'. Then for 60 times 60, (or 3600) this he called a *Sexagena secunda*, (or second *Sexagesimal* Integer,) and marked any Number of them less than 60, by the Signs of Numbers less than 60, with two Dashes: Thus, 60 times 60 (or 3600) was marked I''; two times 3600, thus II''; ten times 3600, thus X''; and so on to 59 times 3600. In this manner the Notation went on: And when a Number less than 60 was joined with any of these *Sexagesimal* Integers, their proper Expression was annexed without the Dash: Thus, the Sum of 4 times 60 and 25 is expressed thus, IV',XXV. The Sum of twice 60, ten times 3600, and 15 is expressed X'',II',XV; the highest *Sexagesimal* being set next the Left-hand. As for the *Sexagesimal* Fractions, they were marked the same way, their Numerators by the Signs of Numbers less than 60, and their Denominators by one or more Dashes (according as they were Primes, Seconds, &c. *i. e.* 60, 3600, and so on in the order of the Powers of 60) set either over the Numerator upon the Left-hand, or under it upon the right. Thus five sixty Parts are marked 'V or V,. And fourteen 3600 Parts 'XIV or XIV,. The Practice by this Notation would be easier than their common Method; yet still very difficult, especially in Multiplication and Division, as appears by the Work of *Barlaamus Monachus*, called *Logistica*; wrote in *Greek* about 1350; translated into *Latin*, and published 1600. And here it is remarkable how very near this Method is in the general Nature of it to the *Arabick*. He wanted no more, but instead of *Sexagesimal* Progression, to make it Decimal; to make the Signs of Numbers from one to nine simple Characters; and lastly, to make a Character which signifies nothing by itself, serving only to fill up Places. But every Age and Nation has its Genius; and therefore we owe this to others.

As this *Sexagesimal* Notation was used chiefly in the Astronomical Tables, so for the sake of these, it was not laid aside immediately after the Introduction of the *Arabick* Notation. The *Sexagenæ Integrorum* went first out; but the *Sexagesimal* Fractions continued till the Invention of the *Decimals*. *Regiomontanus* about the Year 1464, is the first we know who in his *Triangular Tables* divided the Radius into 10,000 Parts instead of 60,000; and so tacitly introduced decimal Parts in place of *Sexagesimals*. *Ramus* in his Arithmetick, written about 1550, (and published by *Lazarus Schonerus* in 1586) uses decimal Periods in carrying on the Extraction of Square and Cube Roots to Fractions. The same did our Country-men *Bucklæus*, before *Ramus*; and *Record* about the same time. But the first who wrote an express Treatise of Decimals, was *Simon Stevinus*, about 1582.

As to the *Circulating Decimals*, Dr. *Wallis* was the first among us who took much notice of them. But I have spoke of this already.

Another most wonderful Improvement that the Art of Computation has received from the Moderns, is the *Logarithms*; the unquestionable Invention of the Lord *Neper*, Baron of *Merchiston* in *Scotland*, towards the end of the sixteenth Century, or beginning of the seventeenth.

Dr. *Wallis* is the Author of the *Arithmetick of Infinites*; which has been very usefully applied in *Geometry*.

But the Consummation of the Art is in the *Algebraick* Method of resolving Questions: The particular History of which, I have said nothing of; because, tho' *Algebra* belongs to *Arithmetick* in a larger view, yet I have limited myself to Arithmetick taken in a more strict sense, as it is distinguished from *Algebra*: Therefore I shall only say, that most of the Authors mentioned have also wrote upon the Algebraick Art, which came into *Europe* at the same time, and by the same hands, as the Numeral *Notation*: *Lucas de Burgo* being reckoned the first *European* Writer on this Subject.

INDEX, or TABLE of CONTENTS.

ARITHMETICK.

BOOK I.

Of WHOLE NUMBERS.

CHAP. I.

Of the Nature of ARITHMETICK *in general, as to its Object and Operations ; with the Division and Order of the Science.*

§. 1. *DEFINITIONS.*

I. ARITHMETICK is the Science, or Knowledge of Number; which is either *Unity*, or a *Multitude* of Units.

II. *Of* UNITY. When we confider any thing by itfelf alone, either as indivifible, or at leaft undivided; or alfo confidering feveral things as connected in fome certain manner, thereby making up a whole, neglecting what differences may be among them in other refpects; the Idea we have of this thing, or Collection of things, confider'd in this manner, is called *Unity*, or *One*, *i.e.* an individual thing of a particular Kind and Name; as one Man, one Stone, one Kingdom, one Army.

III. *Of* MULTITUDE. When we confider feveral things as really diftinct Individuals, and which feparately taken we would call Units, whether they are of the fame, or of different kinds and natures of things; or whether they are really feparated from one another, or only diftinguifhed by the Imagination, as the conceivable Parts of any continuous Body (for Example, a Rod) looking at no more in them but that they are not the fame individual thing; the Idea acquir'd by this way of confidering them, is called *Multitude* (or *many*, in diftinction from one;) fo we fay, a Multitude of Men, of Horfes, of Trees.

SCHOLIUM 1. The Units, or Individuals that make a Multitude, may either be of the fame Kind, or Species and Denomination of Things, or they may not be fo: for however different feveral things are in Name and Nature, the Idea of Multitude arifing from them

is

is the fame: So, for Example, a Man, a Tree, and a Horfe, make as truly a Multitude, as if they were all Trees, or Men, or Horfes: They are at leaft a Multitude of Beings, or Things; for the Idea of Multitude has no dependence upon the Likenefs of the Things from which it is formed, but only upon their different and diftinct Being or Exiftence, in whatever manner they are connected together, or under whatever other differences they really exift, or are conceived to exift.

Scholium 2. We may alfo conceive Multitudes of things under the Notion of one, or many; and fo we may fay one Multitude, or a Multitude of Multitudes.' But it is to be obferv'd, that in this cafe, the Multitudes which make the Parts of one Multitude, are conceiv'd each as an Unit, or one of its own Species, (viz. Multitude,) to diftinguifh it from the Multitude of which it is one conftituent Part: fo that Multitude in its general Nature is ftill a Collection of Units, which are in all cafes fimple Units in refpect of the Multitude which they compofe; tho' they may be themfelves Multitudes compofed of more fimple Units. And this Diftinction of Units may be very well diftinguifh'd by the Names, *Simple and Collective Units*.

IV. *Of Number.* *Unity* and *Multitude* comprehend the whole Object of Arithmetick, and are both comprehended under one general Name, *Number*; whofe Definition does therefore take in the other two; and may be made thus, *viz. Number* is the Name of that Idea or Notion under which things being confider'd, they are faid to be *One* or *Many*? Every particular *Multitude* having a diftinct Name; as two, three, four, &c. As afterwards will be taught.

Scholium. We muft make a little Stand here, and take notice of an old Difpute among *Arithmeticians* about the Definition of Number; fome denying Unity to be a Number, and others affirming it: about which there has been a great deal of Argument fill'd with abundance of idle and nonfenfical Jargon, to the fhame even of fome late Writers: For after all the learned Contention, it dwindles into a meer Difpute about the Name, or what fhall be the Ufe of the word *Number*; which no doubt each Party has a Right to eftablifh for themfelves at pleafure; but no Right to impofe it upon others: And fo then where is the ground of a Difpute? For if any Man asks me whether Unity is a Number, I muft firft know of him what he calls a Number, and then I anfwer him according to his own Definition; or I firft give him my definition of the word Number, and then anfwer his Queftion out of that. But we fhall hear their different Definitions: Some define *Number* a *Multitude* of Units; and according to them it is plain, *Unity* is not a *Number* in that fenfe in which *Unity* and *Multitude* are diftinguifhed, (for we have obferved already how *Unity* and *Multitude* may be applied to the fame Subject in different fenfes;) fo that thefe by denying *Unity* to be a *Number*, do only deny it to be a *Multitude* in the fame fenfe or application in which it is *Unity*, which no body will affirm. Others adhere to the former Definition which comprehends *Unity* and *Multitude*; but fome of them are as much in the wrong, becaufe they contend about it as if they had the only right to fettle the Ufe of Words; and ftill they are more ridiculous to pretend they are arguing about the Nature of Things themfelves, when it's only about a Word: and if it does not yet appear that there can be no more in the Difpute, let this be confider'd, *viz.* That *Unity* and *Multitude* are agreed upon to fignify different things. And I believe it muft be yielded, that thefe comprehend the whole Object of Arithmetick; therefore *Number* muft either fignify the fame with one of thefe, or be apply'd as a general Name to both; and then the only remaining Queftion will be, Which is moft reafonable? And this, I think, will be eafily decided by confidering, that of two Words merely fynonimous, one is fuperfluous; but it's often very convenient to comprehend feveral things, which have alfo their different Names, under one general Name, becaufe of fome common thing in which they agree, as it is in this prefent cafe. And thofe who would make *Number* equivocal with a *Multitude*, are preffed alfo with this other Difficulty, *viz.* That if they retain their

Defi-

Definition of *Arithmetick*, *viz.* the Science of *Numbers*, then *Unity* will be no part of the Object of *Arithmetick*, fince it is not a *Number.* But this, I believe, they will not fay; for whatever can be any part of the *Data*, or Means by which a Queftion in *Arithmetick* is folved, or be itfelf a real and pofitive Anfwer, muft belong to the Science, as a part of its Object. And indeed, tho' *Euclid* defines *Number* to be a *Multitude* of *Units*, yet all along he treats of *Unity* under the fame Name.

V. Of NUMBERS, *Abftract and Applicate.* 1. When in things number'd we confider their Number, abftracting from (*i. e.* not attending to) their other particular Properties and Differences ; the Idea or Notion we hereby form of Numbers, is called *abftract* or *general* ; or, we are faid to confider Number *abftractly :* Becaufe whatever is true of that Number of things confider'd fimply and purely in the Number, is true of the fame Number wherever it is found, or in whatever things it exifts.

SCHOLIUM. We can form no Idea of Numbers, without that of things number'd; becaufe it is an Idea form'd by comparifon of things : Yet while we confider a Number of particular things, tho' we ftill know that the Number is infeparable from other Ideas that make up the complex Idea of thefe things, it's in our power to confider and compare only the Numbers of things together, and examine their various Properties and Differences; and the Mind can perceive at the fame time, that whatever is true of the Number of thefe things, muft neceffarily be true of the fame or equal Number of whatever other things, wherein the Number only is what we confider and compare. From whence it is that we fpeak of Numbers without naming any particular things; by barely naming the particular Number, or joining it with the general word Thing, (which is always fuppofed, when not mentioned.) So we fpeak of the Number, Two, Three, &c. *i. e.* two, three things, without pointing out any particular thing : Becaufe where nothing is taken into confideration (as the Subject of Comparifon and Reafoning) but the Number of things, then any things may be fuppofed; and fo tho' the Names *Two* or *Three* are Names of particular Numbers, infeparable from particular things, yet becaufe the fame Numbers in every other thing muft have the fame Names and Properties, we make ufe of the Name without mentioning particular things : not becaufe that Name belongs to (or reprefents) an Idea of that particular Number which is not connected with any particular things; but becaufe it is a general Name applicable to the fame Number of whatever particular things; and is ufed in this manner without mentioning any thing, when it's indifferent which things are fuppofed, (*i. e.* when the Number only is the matter in queftion;) in the fame manner as we have here ufed the word *Number* itfelf, without mentioning a particular Number, as One, or Two, &c. Not as if the word Number reprefented an Idea different from all Particulars; but as it is a general Name comprehending them all.

2. When we confider Number not in its general Nature, as above explained, but as it is a Number of certain particular things, as two Years, two Men, or two Yards ; then we call it an *Applicate* Number: which Name I chufe, for its obvious Meaning, rather than the word *Contract* or *Concrete*, which fome Authors ufe.

SCHOLIUM. When particular things are mentioned, there is always fomething more confidered, than barely their Numbers; fo that what is true when Numbers are compared in their abftract or general Nature, (*i. e.* when nothing but the Number of things is confider'd) will not be true, when the Queftion is limited to particular things : So, for example, the Number Two is lefs than Three, yet two Yards is a greater quantity than three Inches : for the Comparifon here is not fimply of the Number of things two, and three ; but of the Numbers joined with another Confideration, *viz.* that of their lengths. And when things are of quite different Species, then tho' we can compare their Numbers *abftractly*, yet we cannot compare them in any *applicate* Senfe. And this Difference is neceffary to be confider'd, becaufe upon it the true Senfe, and the Poffibility or Impoffibility of fome Queftions depends; as we fhall learn more particularly afterwards.

COROLLARIES.

1. Number is unlimited in refpect of Increafe; fo that beginning at Unity, and adding to it another Unit, and to this laft Collection another Unit, and fo on, we may proceed *in infinitum, i. e.* we can never come to an end, or never conceive a Number, but ftill there is a greater. But on the fide of Decreafe it's limited; Unity being the firft and leaft Number, below which therefore it cannot defcend. In what fenfe Unity is faid to be divided into Parts, which make a greater Number, fhall be confider'd in its place. Alfo we may not only begin at Unity, but at any other Number, and increafe it *in infinitum*, by the continual joining of Unit after Unit; or diminifh it to nothing, by continually retracting or taking away Unit after Unit.

2. Any Number may be increafed by any other Number, or by any Number of Numbers; for every Number is either Unity, or a Collection of Units, which can be joined feparately to another Number till they be all joined. Alfo it may be decreafed by any Number not greater than itfelf; or by any two or more Numbers, which taken all together do not exceed it, *i. e.* fuch a Number or Numbers may be taken out of it.

3. Every greater Number may be confider'd as compofed not only of Units, (which are its moft fimple conftituent Parts) but alfo varioufly of other Numbers (leffer than itfelf, according to the variety of leffer Numbers, whofe Units taken all together make a Collection equal to that Number; or, according to the various Diftributions that may be made of its Units, by putting them together in feparate Collections: where alfo every leffer Number may be conceived as a Part of every greater; which is as a Whole with refpect to all the leffer Numbers, which, being joined together, make up that Number.

§. 2. *Of the general Division, and Order of this Science.*

THE moft general Divifion of Arithmetick is that of *Theory* and *Practice*.

The *Theory*, or Speculative Part, is that Science which confiders and explains the Properties and Relations of *Pure and Abftract Numbers;* confifting of fuch Propofitions as exhibit to the Underftanding certain Truths concerning Numbers, either more general or more particular; as *Axioms* and *Theorems*.

The *Practical* Part is the Art of Numbering, or applying the Theory to the Solution of Queftions, either in abftract or applicate Numbers; confifting of *Problems*, or fuch Propofitions as require fomething to be done or effected: and gives us a Rule for the Performance; teaching how, by means of certain known Numbers, to difcover other Numbers connected and related to them, according to the Conditions propofed in the Queftion; or at leaft to find that from the given Numbers, compared and applied to one another, as the nature of the Queftion requires, there arifes no Number.

Obferve, Some confider as Theory all that is propofed in abftract Numbers, whether Theorems or Problems; and the Application to Queftions in applicate Numbers only, they call the Practical Part. Others define the Theory as above; and confine the Practical Part to Problems of Abftract Numbers: and Problems of Applicate Numbers they call the *Effective* Part.

As the Truth and Reafons of the Practical Rules are contained in *Theorems*, with other more general Principles, as *Definitions* and *Axioms*, fo they are to be reckon'd Deductions from them, or rather their Applications: And therefore in the natural Order, the Theory ought to precede the Practical Part. But yet thefe two Parts ought not, and cannot be treated entirely feparate from one another; *i. e.* all of the firft Kind together, and afterwards all of the other: But they muft be mix'd together according to their Dependence. It's certain that *Theory* muft precede *Practice*, becaufe that contains the Grounds and Reafons of this; yet 'tis as true, that we can make but a fmall Progrefs in Theory, till we
under-

underſtand the fundamental Elements and Rules of the Practical Part: Theſe being indiſpenſibly neceſſary both for underſtanding the Senſe, and illuſtrating, or proving by Examples, the Truth of Theorems.

Wherefore the *Diviſion* that muſt be followed in explaining this Science, is not that of *Theory* and *Practice*, (tho' theſe muſt alſo be diſpoſed according to Reaſon, and their natural Connection and Dependence;) but the moſt proper and reaſonable *Diviſion* is into two other Parts, under the Titles of the *Simple* and *Comparative Elements*.

If we reflect upon the Definitions and Corollaries already explained; theſe are ſo many of the firſt general Principles and Axioms of the Science: And from theſe we ſhall eaſily underſtand the Reaſon of this Diviſion, and what in general belongs to each Part. For it's plain, that the moſt general, and what we may call the only *abſolute* Property of Number, is, *a Capacity of Increaſe* in infinitum, *or Decreaſe to nothing:* all other particular Properties are *relative*, depending upon the Compariſon of *Numbers* together. And ſince there is nothing in *Numbers*, but different Collections of Units, or different Compoſitions of leſſer *Numbers* in greater, theſe particular Properties muſt all depend upon the Effect of different Applications of *Numbers* to one another, whereby they are variouſly compounded together or reſolved, according to certain Conditions. And for Arithmetical Problems, or Queſtions, in which an unknown Number is to be found by means of certain Connections and Relations it has to ſome known Numbers; theſe Connections can conſiſt in nothing elſe but this, *viz.* That the Number ſought *is* the Reſult of variouſly increaſing and decreaſing the known Numbers by one another, according to the Conditions propoſed; ſo that all that can be known or done in Arithmetick does evidently relate to, and depend upon, the Application of Numbers to one another by Compoſition and Reſolution, or Increaſing and Decreaſing them.

Therefore the firſt and fundamental Part of Arithmetick is the Knowledge of the various Rules and Operations (with the Principles upon which their ſeveral Reaſons depend) by which Numbers are compounded and reſolved; *i. e.* increaſed and decreaſed by one another, which are the fundamental Elements of Practice; including in general all that can be done with Numbers; and indiſpenſibly neceſſary alſo for underſtanding and proving the more particular Theory; which does all relate to the Effect of theſe Operations: Which I have therefore juſtly, I think, conſider'd as the *Simple* and Primitive Elements of Arithmetick. What further Subdiviſion of this is neceſſary, ſhall be ſhewn in its proper place.

All the reſt of the *Science of Arithmetick* I comprehend under the general Name of *Comparative Elements*; becauſe it conſiſts of ſuch relative Properties as ariſe from the comparing of Numbers together, and applying them to one another by the various Methods of Compounding and Reſolving, taught in the firſt Part; as alſo the Solution of ſuch Queſtions as depend upon theſe Relative Properties.

In the remaining Part of this Book, with the ſecond and third, you have the firſt Branch, or Simple Elements explained; and the Comparative Elements in the remaining Books.

§. 3. *Of the Operations of* ARITHMETICK *in general.*

BY what has been already explained, it will be obvious, That all the *Operations* in Numbers are of two Kinds in General, *viz. Augmenting* and *Diminiſhing.* Each of theſe are performed after two different ways, and thereby come under two different Names: Thus, *Augmenting* is divided into ADDITION, and MULTIPLICATION; *Diminiſhing* into SUBSTRACTION, and DIVISION: Which ſhall be explain'd in order. Some add a third Claſs, whoſe Branches are call'd *Involution*, and *Evolution*; or alſo *Raiſing of Powers*, and *Extracting of Roots.* But theſe may be comprehended under *Multiplication* and *Diviſion*; for the Operation is of the ſame general Kind, only under certain Limitations. They will deſerve however to be explain'd diſtinctly by themſelves.

But

But there is yet fomething previous to all this, *viz.* The Knowledge of the Signs whereby our Ideas of Numbers are expreffed or reprefented; for without fome eafy Method and Art of reprefenting them, fo as they may be clearly and diftinctly compared, there could be little or nothing known or done in Numbers. This Art we call the *Notation* of Numbers; which, if we confider by itfelf, and in its primary Defign, is only a neceffary Inftrument for the better and more eafy Comparifon of Numbers, and performing their Operations; and is therefore rather a Handmaid, than an effential Part: For we can call nothing effential but what belongs to the very Nature of a Thing; and which being taken away, the thing would be deftroy'd: not that which is arbitrary, and may be changed, and another thing put in its place at pleafure, as it is in the Notation of Numbers. However, as there muft firft be a Method of Expreffion inftituted, whatever that is, fo far as the Rules of Operation depend upon it, it is the Foundation of them; and therefore it is commonly look'd upon as the firft Rule or Operation in Arithmetick, making in all five fundamental Operations, *viz. Notation, Addition, Subftraction, Multiplication,* and *Divifion.* But ftill it ought to be confider'd only as an arbitrary Rule and Foundation, which requires and fuppofes no other Principle but this, *viz.* That *any of our Ideas may be reprefented by any Marks or Signs we pleafe to inftitute :* whereas the other Operations; befides what they owe to the Notation, have alfo a dependence upon Reafonings from the Nature of Numbers themfelves.

Before we enter upon thefe Operations, we muft here repeat an Obfervation which has been already made, *viz.* That all Science muft begin with *Theory* as a Foundation for *Practice.* Now this Order we have in effect followed; for the Definitions and Corollaries explained in §. 1. are the firft and more general Principles of this Science; to which if we join this general *Axiom, viz.* That *the Whole is equal to all its Parts,* we have all that is neceffary for entring upon thefe practical Elements. What other Principles are employ'd in particular Rules, fhall be explain'd in order as we go on; for they are gained moft part by confequence in the progrefs of the Science.

CHAP. II.

Of the Notation, or Expreffion of NUMBERS, *with their Diftinction into Integral and Fractional.*

§. 1. *Of the Notation of Numbers.*

DEFINITION.

NOTATION *is the Method or Art of Expreffing Numbers :* which is done two ways; by certain *Words* or *Names,* and alfo by certain *Signs* or *Characters,* called *Figures*; the one correfponding to the other in the Reprefentation of the fame Number, and both equally neceffary: The Figures being contrived for the eafy management of Operations, whereby the greateft Numbers are compared, and the Operations performed with the greateft eafe and readinefs; without which, our Knowledge in Numbers had reached a very fhort way. And fo much does the Science owe to thefe, that upon this Account fome call it, *The Art of Figuring*; but we might better call it, *The Science*
of

of Numbers, as they are reprefented and managed by Figures: So that this is the principal Branch of *Notation;* which yet cannot be without the other, the Names of Numbers being neceffary for our converfing or fpeaking to one another about Numbers, and the Refult of the Operations made by the Figures, and for the actual numbering or telling over things, by the Application of Names in an orderly Progreffion from Unity, ftill joining one Unit after another to the Collection; or telling things out one by one. For without fome Signs whereby the Number, as it increafes, is continually diftinguifhed, we could make nothing of the Numbers of things, nor compare one Number with another: and Words are the moft proper and convenient Signs for this purpofe; which are alfo of good and neceffary Ufe in making the Operations with the Figures.

The Syftem of the Names of Numbers is indeed a part of our Language, and therefore the Writers on Arithmetick fuppofe them to be known, and reckon it their bufinefs only to explain the Reprefentation of Numbers by Figures, and their Correfpondence to the Names: But it will be a more regular and juft Method to explain both the Syftems of Names and Figures by themfelves; and then fhew their mutual Correfpondence. Thefe two Syftems I fhall explain under the Titles of the *Nominal* and *Figural Notation;* which being compared, their mutual Correfpondence will be eafily underftood. But this I fhall more particularly explain in the Solution of a *Problem,* teaching how from the Expreffion of any Number in one manner, to find its correfpondent Expreffion in the other.

(1.) Nominal Notation, *or the Expreffion of Numbers by Words or Names.*

A DIFFERENT fimple Name for every Number, or even for as many particular Numbers as we have occafion to confider in human Affairs, would be a Burden altogether infupportable; but it is more happily contrived, that a few fimple Names, and thefe compounded together in a very eafy manner, anfwer all the Ends and Purpofes both of Speculation and Practice.

The Simple Names of Numbers are thefe,

One,	Ten,	*Explanation of this* TABLE.
Two,	A Hundred,	*One* is another Name for *Unity:* the reft of the Names
Three,	A Thoufand,	to *Ten* exprefs Numbers in a Series from Unity, by the
Four,	A Million,	continual joining Unit after Unit; fo *Two* expreffes *One*
Five,	A Billion,	more *One*; *Three* expreffes *Two* and *One*, and fo on to
Six,	A Trillion,	*Ten*: a *Hundred* expreffes *Ten-Tens*; a *Thoufand* expreffes
Seven,	A Quadrillion,	*Ten-Hundreds*; a *Million* is a *Thoufand-Thoufands*; a *Billion*
Eight,	*&c.*	is a *Million* of *Millions*; a *Trillion* is *a Million* of *Billions*;
Nine,		a *Quadrillion* is a *Million* of *Trillions*.

So that we have here Names anfwering to the natural Series of Numbers from *Unity* to *Ten.* But after this, the Series is interrupted, and we pafs to the Names of greater Numbers: And all that remains to be explain'd is, how the intermediate Numbers are named from *Ten* to a *Hundred,* and from a *Hundred* to a *Thoufand,* and fo on. Which is done thus, after *Ten* the Names are compounded of *Ten* and the preceding to a *Hundred.* Firft, from *Ten* to two *Tens* we proceed thus;

I   Ten,

Ten,
Eleven,   Ten and One,
Twelve,   Ten and Two,
Thirteen,   Ten and Three,
Fourteen,   Ten and Four,
Fifteen,   Ten and Five,
Sixteen,   Ten and Six,
Seventeen,   Ten and Seven,
Eighteen,   Ten and Eight,
Nineteen,   Ten and Nine,
Twenty,   Two Tens.

(Equal to)

Then from *Twenty* we simply join the Names of the first *nine* Numbers thus, *Twenty-one, Twenty-two,* &c. to *Twenty-nine:* then the next Number is *Three Tens,* called *Thirty*; and the same Composition of Names we use from *Thirty* to *four Tens,* called *Forty :* and from this to *Fifty,* (or *five Tens:*) and so on to *Sixty,* (or *six Tens :*) *Seventy,* (or *seven Tens :*) *Eighty,* (or *eight Tens :*) *Ninety,* (or *nine Tens :*) , and after *Ninety* and *nine, One* added makes *ten Tens,* or a *Hundred.* Then from a *Hundred* (or *one Hundred*) we proceed by joining with it all the preceeding Names from *One* to *Ninety* and *nine,* thus, *One Hundred* and *one, One Hundred* and *two,* &c. to *One Hundred* and *Ninety nine,* to which one added makes *Two Hundred.* In the same manner we proceed from *Two Hundred,* to *Three Hundred*; and so on to *Nine Hundred Ninety* and *nine,* and then *one* added makes *Ten Hundred,* or *One Thousand.* In like manner we proceed from *One Thousand,* joining with it all the preceding from *Unity* till we come to *Two thousand,* and so on to a *Thousand-thousands,* or a *Million*; and with this also we join all the preceding Names from *Unity* to *two Millions,* &c. to a *Million* of *Millions,* or a *Billion*; and so on to a *Trillion* and *Quadrillion.*

Observe, If we would proceed farther, we may use these Names, *Quintillions,* (or a *Million* of Quadrillions;) *Sextillions,* (or a *Million* of Quintillions,) and so on; filling up the intermediate Numbers as before : But for any real Use in human Affairs, we need no Names above Millions. In Mathematical Work, greater Numbers occur; but they are manag'd by Figures, and can be compared without Names; which, if required, may be contriv'd in the manner now mention'd. *Observe* also, that some instead of the simple Names Billions, Trillions, Quadrillions, *&c.* chuse the compound Names *Millions of Millions; Millions of Millions of Millions; Millions of Millions of Millions of Millions*; and so on, compounding the word Million once more gradually : But the other simple Names seem more convenient, tho' the contrivance and way of making the complex Names is more obvious and easy; however, since we have little use for Names above Millions, we need not dispute about the difference.

(2.) FIGURAL NOTATION.

AS a few simple Names serve all our Purposes in Arithmetick, so yet fewer simple Figures are found sufficient, not only for common Use, but even to carry us thro' the Infinity of Number: Which Figures, with their corresponding Names, are these;

Figures	0	1	2	3	4	5	6	7	8	9.
Names	Nought	One	Two	Three	Four	Five	Six	Seven	Eight	Nine

That a Figure signifying of itself *Nothing,* (or no Number) is necessary, we shall presently see. In the mean time *observe,* that the Number of simple Figures being *Ten,* they are hence called the Digits, from the Number of *Fingers* (*Digiti*) on our Hands. How this Number came to be chosen, we shall afterwards consider.

All other Numbers greater than Nine, (9,) are expressed by Combinations of these Digits; placing them together in a Line in various Orders; every Figure changing its Value according to the place it stands in. By this

GENERAL RULE.

In a Rank of Figures placed together in a Line, (reckoning the Order of Places from the Right to the Left-hand; that being the first which is first on the Right-hand; and the second, third, &c. being in order from that on the Left) any Figure in the first place represents the primitive simple Value above expressed, as if it stood alone; and in every place gradually towards the Left-hand, it signifies ten-times as many as it would do in the preceding place.

Thus 1, 2, 3, &c. in the first place signify simply so many Units as above named; but in the second place of a Rank, any one of them signifies ten times so many Units, (or so many times ten.) *Ex.* 48 signifies forty and eight, [*i. e.* four tens for the value of 4 in the second place, and 8 in the first place.] Again, 60 signifies sixty, [or six tens, which is the value of the 6 in the second place; the o in the first place signifying nothing of itself.] Again, in the third place they signify ten times their Value in the second place; or ten times so many Tens, (*i. e.* so many hundreds:) and so on gradually towards the Left-hand, still increasing their value tenfold of that of the preceding place. So 600 is six hundred; 642, is six hundred, forty and two.

Now this general Institution being well conceiv'd, it will, I suppose, be found evidently sufficient for the expression of all Numbers; or that all Numbers in a gradual Series from Unity may be represented thereby. And if this is not evident, it may be made so—Thus; Every Number is either less than Ten; which is expressed by one of the nine significant Digits: or it is a Number of Tens less than ten; which is therefore expressed by some Digit in the second place, and o in the first: or it is such a Number joined with a Number less than ten; which last Number is expressed by a Digit in the first place: or it is a Number of Ten-tens (or hundreds) less than ten; which is therefore expressed by some Digit in the third place, and o in the first and second places: But if with this Number is joined any lesser Number, (*i. e.* any of the preceding Classes) that is expressed by Digits in the first or second, or in both places. The Progression to greater Numbers, I think, will be now plain enough. Or we may consider it in this manner—Any Combination of Figures expresses some one determinate Number, according to the instituted Value of Places; and if Unity is joined to this Number, the Sum can be expressed: for this is done by changing the Figure in the first place, and taking the next greater, *i. e.* for 1, taking 2; for 2, taking 3, &c. and if 9 is already in the first place, then because nine and one make ten, we must (according to the Institution) put o in the first place, and change the Figure of the second place, taking the next greater; and if that is also 9, we put o also in the second place, and change the Figure of the third; and so on, till we come to a place in which there is a Figure less than 9. So that if all the Figures of the given Example are 9's, we set o in all their places, and set 1 on the left of all. *Example* I. 47 and 1 is 48. *Ex.* II. 29 and 1 is 30. *Ex.* III. 499 and 1 is 500. *Ex.* IV. 999 and 1 is 1000.

Now it's plain, that if to any Number we can add one and express the Sum, by the same Rule we can add 1 to the Sum; and 1 to this last Sum, and so on. And because 1 may be the given Number, it follows that we can by this Institution express any Number in the natural Series from Unity *in infinitum*; and the way of doing it is here also made evident.

SCHOLIUMS.

I. We see now that tho' the Figure o signifies nought of itself, yet it is not useless; but indispensibly necessary to fill up places, that other Figures may possess such places as they ought to be in, for the expression of certain Numbers which could not be expressed by this fundamental Rule, without the help of this Character. One Example is enough to shew

it,

it, and we shall chuse the Number *Ten*, which could not otherways be exprefs'd: for by this Inftitution, 1 in the fecond place is ten; but there cannot be a fecond place unlefs there be a firft; and if any fignificant Figure ftand in the firft place, the whole will make a Number greater than Ten: fo 14 is ten and four; therefore the Figure in the firft place muft fignify *nothing*, and ferves only to make two places, that the 1 may be put in the fecond, where it fignifies Ten: thus, 10.

2. The Figure o ftanding on the Left-hand, or in the laft place of any Rank of Figures is altogether ufelefs; fo 04 is no more than 4, becaufe the Value of Figures rifes from the Right-hand to the Left.

3. According to this Inftitution, each of the nine fimple fignificant Figures may be confider'd as having two Values; the one certain and determinate, known by its Form, which is that it fignifies fimply by itfelf, (as 4 is *four*) and may be called its *fimple* or *primitive* Value: the other is uncertain and variable, depending upon its place in a Rank with others; fo any Figure in the fecond place is fo many Tens; in the third place it is fo many Ten-tens, (or hundreds) and fo on in a ten-fold Increafe: and this may be called the *fecondary* or *local* Value, *i. e.* the Value of the *Place*; and thefe two Values compounded (or the one repeated as oft as the other contains Unity) makes the compleat Value of that Figure in that place. For Example, 4 in the fecond place is four-times Ten, (or forty;) and in the third place it is four-times Ten-tens, or four hundred. But in the firft place (and in no other) thefe two Values coincide; for here there is no Value but the *Simple*.

4. If thefe two Syftems of *Names* and *Figures* are duly compared, their mutual Correfpondence will be eafily underftood; fo that you'll find little difficulty in exprefling any Number in the one manner which is firft exprefled in the other. But that no body may complain, I fhall explain this more particularly in the following *Problem*.

PROBLEM.

Having any Number exprefled by Figures, *how to read or exprefs it in* Words; *or, having it exprefled in* Words, *how to write or exprefs it in* Figures.

CASE 1. *From a given Expreffion of any Number in* Figures, *how to read it in* Words.

We have already confider'd how that any Number being exprefled by Figures, every Figure may be confider'd in a double view, *i. e.* according to its *fimple* and *local* Value. Again, the Syftem of Names as above explain'd, is fo contrived, that Ten times Ten; Ten times Ten times Ten; Ten times Ten times Ten times Ten; [and fo on, which are the Value of places after the fecond, which is Ten] have diftinct Names, either fimple or compound; and fuch Values taken any number of times lefs than Ten, are named no other way than by exprefling the number of times (lefs than Ten) that Value is repeated; fo that if we know the Names of both values for every Figure in any Rank (or Expreffion of a Number, which has more than one place) and exprefs each according to the Compofition of thefe two Values, then we have the Expreffion fought, for the whole Rank or Number propofed. The Names of the fimple Values of Figures we have already learnt; what remains is to know the Names for the local Values, which is the Defign of what they commonly call the Table of Notation (or Numeration) which I fhall put in a more convenient Form.

TABLE

TABLE of NAMES *for the Local Values of Figures (in which* X *stands for Ten ;* C *for Hundred ;* Th *for Thousand.*)

C, th, Bill.	X, th, Bill.	th, Bill.	C, Bill.	X, Bill.	Billions.	C, th, Mill.	X, th, Mill.	th, Mill.	C, Mill.	X, Mill.	Millions.	C, Thousands.	X, Thousands.	Thousands.	Hundreds.	Tens.	Units.	Names
18th	17th	16th	15th	15th	13th	12th	11th	10th	9th	8th	7th	6th	5th	4th	3d	2d	1st	Places

If you would carry this Table farther, it is to be done by the Names *Trillions,* &c. (already explained) the same way used, as here *Millions* and *Billions* are.

Now the Use of the Table is plainly this. Apply each of these Names in Order, to the several Figures of a Rank, according to their Places (the Name Units signifying only, that the Figure in that Place is taken in its simple Value) and prefix the simple Value, making a Composition of both (observing how any simple Value compounded with Ten is contracted, *viz.* For two Tens, say Twenty, and so on to Thirty, Forty, &c. as above explained) and thus read all the Figures from the Left to the Right. Observe also, that where any Word belongs to the *Names* of *several Places,* which stand all next together, as do the Words, Thousand, and Million, and Billion, that needs not be repeated; but only once expressed, after all the Places to which it belongs are read according to what is more in the Name, as the following Examples will make plain.

EXAMPLES.

86, is Eighty Six

254, is Two Hundred, Fifty, and Four.

7408, is Seven Thousand, Four Hundred, and Eight.

67040, is Sixty and Seven Thousand, and Forty (*i. e.*) Sixty Thousand, Seven Thousand, and Forty.

46258300, is Forty Six Millions, Two Hundred and Fifty Eight Thousand, and Three Hundred, (*i. e.*) Forty Millions, Six Millions, Two Hundred Thousand, Fifty Thousand, and Eight Thousand, and Three Hundred.

24800540362, is Twenty Four Thousand and Eight Hundred Millions, Five Hundred and Forty Thousand, Three Hundred Sixty Two, (*i. e.*) Twenty Four Thousand Millions, and Eight Hundred Millions, &c.

But this may be made yet easier, by considering the several adjacent Places that have any one or more Words common in their Names; from which we have this other

TABLE.

th-Bill.	Bill.	th-Mill.	Mill.	Thousd.	Un.
c : x : Un.	c : x : Un.	c : x: Un.	c : x : Un.	c : x : Un.	c : x : Un.

Observe, If you neglect the Word Un. (or Units) which stands first in the upper Line of this Table; and where the Word Unit stands after the three first Places in the under

Line, put the Name that ftands over every Repetition of c : x : Un. repeating it alfo with the other two, *viz.* c : x. then you make the fame Series of Names as in the preceding Table. And for the Advantage of this Form, we fhall prefently fee it.

The Application of the TABLE.

Begin at the Right Hand of any Number, and divide or feparate the whole Figures into Claffes or Periods of three Figures each, as long as there are as many: then each Period muft firft be read as if it were alone by *Units, Tens, Hundreds,* i. e. the firft place on the right of the Period by its fimple Value, the fecond place as fo many Tens, and the third place fo many Hundreds as the fimple Value expreffes [which Practice is expreffed by thefe Signs c : x : Un. conftantly repeated in the under Line of the Table.] Again, becaufe Figures increafe their Value towards the left, the fame Figures in different Periods are of different Values, and therefore every Period has a common Name that compleats the Expreffion; which are thefe in the upper Line of the Table; (in which *Units* is made the Name of the firft Period merely for a diftinction, but is never expreffed) therefore to apply this Table more eafily, get the Names of the Periods by heart backwards and forwards; and then applying them to the Periods from the right to the left, that you may find the Name of the higheft Period in any Example; and remembring exactly the Names of Periods in order from the left; you muft begin from the left and firft read the Figures of each Period, as c : x : Un. then join the Name of the Period, which is fuppofed to be applied to each Figure of it.

Obferve, That in fome Cafes you'll have a broken or incomplete Period of one or two Figures next the Left-hand, which muft be read by itfelf, juft as it happens to be; but the reft will each have three Figures: yet thefe may be alfo in a true fenfe but broken Periods, i. e. not have fignificant Figures in all their places, as in this 24,048. Again *Obferve,* That in the Periods of Mill. and th-Mill. becaufe the word Mill. is common, you need not repeat it: but in the th-Mill. ufe only the word Thoufand, fuppofing the word Mill. which you need only exprefs after the Figures in the Period of Mill. unlefs that Period be fill'd with o's, as here, 24,000,468,350. which is 24 th-Mill. *&c.* but in this, 24,360,579,200, I read 24 th. 360 Mill. *&c.*

See thefe few more Examples.

278,307,000		278 Mill. 307 Th.
348,026,000,123		348 Th. 26 Mill. 123.
7,200,809,867,345	read thus	7 Bill. 200 Th. 809 Mill. 867 Th. 345.
326,009,478,205,723		326 Bill. 9 Th. 478 Mill. 205 Th. 723.

CASE 2. *Any Number being expreffed in* Words, *to write it down in* Figures.

If the preceding Table of Names correfponding to the feveral places of a Rank of Figures be confider'd, there can be no difficulty in this part of the Problem, which is but the reverfe of the former; yet perhaps it may not be ufelefs to fome to point them out a Method, which is this——Remember exactly the Names and Order of the Periods from Left to Right; and of the three places in each Period: Then obferve what is the Period firft named in the Example, alfo what is the Number (*viz.* of c : x : Un.) applied to that Period; fet down that Number, with a Point after it: then confider what is the Name of the next Period below that in order, and what Number is applied to it in the Example; fet down that Number on the right of the former, with a Point after it; and whatever be the Number, it muft be fet down fo as to poffefs three places, by fetting o in thofe places of the Period to which no Number is applied; (except the firft or higheft Period where this is not neceffary:) fo that if there is no Number applied to any whole Period, (after that which is the higheft in the Example,) fet three o's in its

three

three places: thus proceed to the loweft Period or firft on the right. *Obferve* alfo, that if you find the Name Thoufand mention'd before Million or Billion, it fignifies Th-Mill. or Th-Bill. and is to be diftinguifhed from the Period of fimple Mill. or Bill. as in the preceding Table. One or two Examples will fufficiently illuftrate this Practice.

Ex. 1. To write in Figures this Number; *Forty Million Two Hundred Thoufand and Eight,* I proceed thus; *Million* is the higheft Period; and *Forty* the Number applied, which I write down thus, 40; the next Period is *Thoufand,* and the Number here applied is *two Hundred,* which I fet on the right of the former, thus 40,200; then follows the Period of the *Units,* and the Number here applied is only *Eight;* which, according to the Rule, I fet on the right of the former thus, 40,200,008.

Ex. 2. *Twenty-four thoufand and fixty-two Millions.* Here the name *Thoufand* ftanding before *Million* fignifies *th-Mill.* which is the higheft Period in the Example, and the Number applied is *Twenty-four,* thus written 24; then follows the Period of *Millions,* and the Number is *Sixty-two,* joined to the former thus, 24,062; and becaufe there is no more in the Example, and yet there are two Periods remaining, *viz. Thoufands* and *Units,* I fill them up with o's, thus, 24,062,000,000. For without thefe, the other Figures could not exprefs above Thoufands. If you compare the Examples given for the firft part reverfely, you have enow for this laft Purpofe.

COROLLARIES to the FIGURAL NOTATION.

1. Of two Numbers, expreffed by Figures, (by the Rule and Inftitution explain'd) that which has feweft Figures (or Places) is the leaft Number. *Examp.* 99 is lefs than 100. For tho' the Figures of that which has feweft be all the greateft poffible, (*i. e.* all 9's) and thofe of the other be all the leaft poffible, (*i. e.* 1 in the higheft place, and the reft o's) yet the other will want at leaft 1 to make it equal to this; becaufe 10 is equal to 9 and 1, and 100 equal to 99 and 1, and fo on. Hence the value of an Unit in any place of a Rank is greater than the value of all the preceding Figures on the Right-hand; becaufe taking that Unit in its true value, it is a Number having one place more than are upon its Right-hand.

2. If two Numbers expreffed by Figures have an equal Number of places, that is the leaft Number which has the leaft Figure in the higheft place; or in any other place, all the preceding places on the left being equal. *Ex.* 199 is lefs than 200; and 232 is lefs than 233; and 469 is lefs than 472.

3. If one Number is greater than another, and if you fet before each of them an equal Number of whatever Figures, that which was greateft before will ftill be the greater.

4. If before, or on the Right-hand of any Rank of Figures, (or Number expreffed by Figures) be placed any one Figure, the given Rank expreffes thereby ten-times what it did before (or without that Figure;) and if two Figures are fet before it, it expreffes an hundred-times what it did before; and fo on in a ten-fold Progreffion, according to the fundamental Rule. *Ex.* 480 is ten-times 48; and 4800 is 100 times 48, *&c.* And obferve, That with refpect to the raifing the value of the given Rank, it is the fame whether o, or any other Figure be prefixed; it is certain that another Figure will make a greater Number of the whole Rank as it is now encreafed; but will not make that part of it greater, which was the given Rank. So here in 486, the 48 is equal in value to 480, whatever Figure ftands before it; tho' 486 is greater than 484 or 480.

5. Hence again; Any part of a Number taken from the Left-hand, may be valued thus, *viz.* we may confider what Number it makes taken by itfelf; and then confider the place in which its firft Figure (on the right) ftands; and make the *Name* of that place a common Name or local Value to the whole Rank: Thus, all the Figures of any Rank, excluding the firft on the right, or place of Units, being read by themfelves, (as if there were no Figure before them) is fo many Tens; or fuch a Number taken 10 times. Again, excluding the two firft places, the reft read by themfelves is fo many Hundreds, and

fo

fo on. *Ex.* 246873 : here the 24687 is equal to 24687 Tens, (taking 7 as it were in the place of Units) *i. e.* twenty-four thoufand fix hundred and eighty-feven Tens. The 2468 is 2468 Hundreds; the 246 is 246 Thoufands, and the 24 is 24 Ten-thoufands : as the 2 is 2 Hundred Thoufands.

SCHOLIUMS.

1. Both in the General Rule of the Figural Notation, and thefe two laft Corollaries, I have fuppofed this for a Truth, *viz.* That Ten-times, (or a Hundred, *&c.*) any Number is the fame as that Number of times Ten, (or a Hundred, *&c.*) For Example, that Ten-times 7, is 7 times Ten; which is a Truth I believe will be eafily granted : yet it is of that kind that admits a Demonftration; but I fhall refer it till we come to *Multi-plication*, to which it properly belongs.

2. By the Figural Notation now explained, all Numbers above Nine are expreffed in a compound Form; reprefenting either a certain Compofition of the Number Ten, or fuch a Compofition with the addition of a Number lefs than Ten; which are notably diftinguifhed by this method of Expreffion. But there are various Degrees of the Compofitions of Ten; for a Number compounded of Tens is either a Number of Tens lefs than Ten, which is expreffed by a Digit in the fecond place and o in the firft, as 10, 20, *&c.* or it is fuch a Number taken ten-times, which is expreffed by a Digit in the third place, and o's in the fecond and firft, as 300, 400, *&c.* or fuch a Number as the laft taken ten-times, as 2000, 5000, *&c.* and fo on. All which different Degrees are expref'd by a Digit in different places. From which it's clear, that if feveral of thefe degrees be joined together, *i. e.* if there is a Rank confifting of more than one fignificant Figure, and having at leaft one o on the Right-hand, as 460, which is 46 Tens, that expreffes a Number which is alfo a Compofition of Tens; but if there is a fignificant Figure in the place of Units, that Number contains fo much odd or over a certain number of Tens, as 468, which is 46 Tens and Eight. Now there are two things to be remarked from this Method of Notation : Firft, That every Number is diftinguifhed by the very Expreffion into as many parts as there are fignificant Figures in it; each of which is expreffed feparately by fetting as many o's before it as there are Figures before it in the given Expreffion; as in the following Example. And fecondly, That each of thefe Parts is a certain Compofition of the Number Ten (as above explained) except that which is in the place of Units, which is always lefs than Ten. *Thus*, the Number 4682 is equal to 4000 and 600 and 80 and 2, which is more fimply and conveniently written all in one Rank 4682; whereby all the Parts are as clearly and intelligibly marked out by the meer fituation of the Figures, (according to the Rule.) Now tho' this is really a compound Form, expreffing feveral Numbers diftinctly from one another; yet becaufe it is the moft fimple way of expreffing that Number, or Term in the natural progreffion, which is equal to the Sum of all thefe leffer Numbers (according to the Inftitution) therefore it is faid to be a fimple Expreffion of one Number, in comparifon of the other ways of expreffing the Parts feparately; or of the Expreffion of a Number by any other of its component Parts, feparately written each in their moft fimple Form, (in each of which therefore there are Parts, or Figures that have the fame local Value) as if inftead of 8 we fhould write 5 more 3, (which together make 8) or inftead of 74 we fhould write 43 more 31, which therefore in diftinction from the other we may call complex Expreffions reprefenting feparately two different Numbers, the leffer of which has Parts of the fame local Value with the greater. Or we may explain this diftinction of fimple and complex Expreffions, thus : That Expreffion is called fimple, or one Number, which is one of thofe whereby the natural Series of Numbers is expreffed in a continued Progreffion from Unity; but when two different Expreffions of that Series are feparately written in two diftinct Ranks, they are faid to be two different Numbers: and with refpect to that Number to which they are both together equal, they are faid to be a complex Expreffion

of

of it; especially with some Mark or Word betwixt them to signify their being joined together. For Example, 46 is a simple Expression of one distinct Number in the natural Series: But 32 and 14 are two Numbers, which being both together equal to 46, therefore 32 more 14, is a complex Expression of the Number 46. Again, in the same manner any Number which is equal to the difference of two Numbers may be complexly expressed by these two, with a Mark of Substraction betwixt them: As for Example, since 32 more 14 is equal to 46, therefore if 14 is taken out of 46, the remainder is 32; and hence 32 may be expressed thus, 46 less 14.

The same things are applicable to the Parts by which any Number in its simple Form is expressed; as if for 46 we should write 40 more 6; or for 6 we should write 46 less 40.

In the last place observe, That the compound Form of this Notation is the Perfection of it; not merely as it is compound, but the particular Manner of it; because of the certain relation that each part has to one Number, *viz. Ten*, by the constant, regular and uniform Progression in the Value of Places: For by this Similarity or Likeness of some Parts in different Numbers, (*i. e.* of Parts expressed by Figures in the same or like places;) and the Connection or Relation of all their Parts by means of the common Number Ten, to which they have all a relation; Numbers can be compared together, and their Operations performed in a very easy, clear and distinct manner. And more particularly we have from the Method of Notation this Principle of Operation, *viz.* That some Operations may be performed by the few simple Characters, taken and applied in their primitive Values; and what is wrong or deficient by the neglect of their local Values may be again made up, by the order or due placing the several Figures arising from the Operation: And in other cases where we cannot work by single Figures, yet we can take the Parts of Numbers, (*i. e.* two or more of their Figures) and consider them in the Value they would have by themselves; correcting the Defects the same way as now mentioned. Which Principle we shall see applied in the following Rules; and here only I shall further observe, That in this it is that the Rule of Notation is the ground of all other Operations; affording regular and easy Rules for expeditious and certain Work, as we shall learn.

As to the History and Invention of this admirable NOTATION, see what is said in the *PREFACE.*

3. We might here make Comparisons betwixt this Method of Notation, and others that have been or may be used, in order to shew the Excellency of this Method; but this will be better understood after you see the Application of it in the Operations, where I have allotted a Place for some general Reflections upon these Operations. However, I shall here explain the *Notation of Numbers* instituted by the *Romans;* which will be proper, because it is also practised for Marking of Chapters and Sections, and such things; and this being compared with what we use, the difference in favour of our Method will be sufficiently evident.

Of the ROMAN NOTATION.

The Characters whereby the *Romans* marked Numbers were taken out of their Alphabet of Capital Letters; thus:

The Simple Characters.

I.	V.	X.	L.	C.	D.	M.
Equal to 1.	5.	10.	50.	100.	500.	1000.

The intermediate Numbers betwixt thefe are expreffed by a repetition of the fame; fetting them together in a Line, thereby expreffing the Sum of all their Values (above expreffed) joined to one another, (for they have no different Values from their Places) the Characters of greateft Value being fet next the Left, as for Example; II is 2. III is 3. VI is 6. VII is 7. XI is 11. XV is 15. XX is 20. LX is 60. LXV is 65. DX is 510. DC is 600. DCCCC is 900. DCCCCLXXXXVIIII is 999. Thefe Examples will fufficiently fhew how all the reft are made.

But to prevent too great a Repetition of the fame Characters, they fometimes fet the leffer Character before the greater; and then it reprefented the difference of thefe two, or the effect of taking the one Number away from the other; thus IV is 4. IIX is 8. IX is 9. XL is 40. CD is 400. CM is 900. And when a Number is expreffed by more than two Characters, if any part of it is thus expreffed, it is fit to diftinguifh it from the Characters on the Left of it, by a Point; thus, 140 may be expreffed C,XL, (for CXXXX) and 148 thus, C,XL,IIX (for CXXXXVIII.) Again, 499 thus CD,XC,IX, inftead of (CCCCLXXXXVIIII) which are more convenient.

Again, for Numbers greater than 1000 or M, they are expreffed after the fame manner. But there are other things in their Syftem, both for fome Numbers lefs than 1000, and efpecially for greater, which I fhall briefly explain. Thus, for D or 500 they write I⊃. and then by adding another ⊃ it gradually expreffes ten-times as much; fo I⊃⊃ is 5000. I⊃⊃⊃ is 50000, and fo on. Again for M or 1000 they write CI⊃, and by joining another fuch Mark as C and ⊃ one on each hand, it expreffes ten-times as much; fo CCI⊃⊃ is 10000. CCCI⊃⊃⊃ is 100000. But laftly, they had a more convenient way of expreffing any Number of Thoufands, which was by drawing a Line over any Expreffion of a Number lefs than a Thoufand, whereby it expreffed fo many Thoufands: fo $\overline{V}$ is 5000. $\overline{VI}$ is 6000. $\overline{X}$ is 10000. $\overline{LX}$ is 60000. $\overline{C}$ is 100000. and $\overline{M}$. is a Thoufand-thoufands, or a Million, 1000000. $\overline{MM}$ is 2000000. But I fhall infift no more.

If we now compare this Method of Notation with ours, it prefently appears by the preceding Examples, that fome Numbers are more fhortly expreffed by the *Roman* Way; but thefe are very few in refpect of what are otherwife: And then there is here no fuch regular Progreffion in the Value of the fame fimple Characters as in the other Method. But we muft learn the worth of this from its Application.

Of the UNIVERSAL NOTATION.

By this Name is not meant any conftant or eftablifhed Method ufed every where for the Expreffion of particular Numbers; but a Method of reprefenting any Number indefinitely, in order to the more eafy and general Expreffion and Demonftration of certain Truths in Numbers, which tho' they be limited to particular Conditions, yet not to particular Numbers, but extend to all Numbers, wherein the fame Conditions are found.

The fundamental Principle of this Notation is the fame as the laft, *viz.* That any Mark or Sign may be inftituted for the Reprefentation of any of our Ideas; and here it is done by Letters, making the fame Letter ftand indifferently for any Number, upon this Condition, That through the fame Propofition and Demonftration it be fuppofed to ftand for the fame Number; *i. e.* when particular Examples are applied, we muft apply the fame Number always to the fame Letter. Thus the Letter A or B, or any other, may reprefent any Number we pleafe to fuppofe.

Obferve, This Notation, and the confequent Operations are called *The* LITERAL or SPECIOUS *Arithmetick*, which is in part the Foundation of the *Algebraick Art*; the defigned Ufe of which in the following Work is to make eafy and univerfal Demonftrations.

For when any Truth is propofed which is not limited to particular Numbers, but only to certain Conditions; it is not a fufficient Demonftration to fhew that it holds in one, or
any

any Number of particular Examples; it muft be fhewn that it will hold good in all Cafes poffible: and as this muft be done by an univerfal Method of Reafoning abftracted from all particular Examples; fo there muft neceffarily be an univerfal Notation for Numbers. It is true indeed, that the Univerfality of a Truth may, in fome Propofitions, be made to appear through one, or a few particular Examples; but for the moft part this would prove very tedious, and require many Words, which would render the Demonftration more difficult and obfcure; and in very many Cafes could not be done at all. Which makes the *Algebraick* Method of Demonftration neceffary in Arithmetick, (as I have more fully reprefented in the *Preface.*) The Principles and Rules of which, as far as this Work requires, you'll find explained gradually as we proceed.

Before we enter particularly upon the other Rules and Operations of Arithmetick, there is another Diftinction of Numbers muft firft be explained; *viz.*

§. 2. *Of Numbers* INTEGRAL *and* FRACTIONAL.

THIS Diftinction proceeds from the Comparifon of leffer Quantities with greater; and to underftand the Nature of it aright, we muft confider the different Notions of *Parts* and *Whole*; thus, Every leffer Quantity is called a Part with refpect to a greater of the fame Species, which is called an *Integer* or *Whole* with refpect to the leffer.

But there is a more general, and alfo a particular Senfe in which one Quantity is called a Part of another. In the general, no more is meant but that it is a leffer Quantity, which with fome one or more Quantities, alfo leffer, make up a Quantity equal to that other; and into which therefore that Greater may be refolved. But in a more particular Senfe, a Part fignifies fuch a leffer Quantity as is contained a number of times precifely in a greater, (or Whole:) *i. e.* it is one of thofe leffer Quantities, all equal among themfelves, into which any Quantity may be refolved or feparated. Or we may alfo conceive it thus, *viz.* As a leffer Quantity, of which a certain Number joined together makes up a greater Quantity or Whole. Hence fuch a Part is called an Equal or *Aliquot* Part; and the number of times it is contained in the Whole, or the Number of equal Parts contained in the Whole, gives a Denomination to the Part, and is called its *Denominator:* fo if any Quantity is contained in a greater 6 times, it is called a fixth part of it. Now it is in this fenfe only that a Part can be underftood in Arithmetick: for in order to compare two different Quantities together by the means of Numbers, we muft confider them as compofed of fome common Element, or equal Part; by the Number of which contained in each, the Comparifon may be made; and the Value of thefe Quantities with refpect to one another be determined. If the leffer is an *Aliquot Part* of the greater, there is no more to be done; but otherwife they muft both be conceived as compofed of, and reducible into, fome common Element or equal Part; fo that if the leffer is not one of thefe *Aliquot* Parts of the greater, yet it is equal to a greater Number of fuch Parts; [for otherwife they cannot be compared together; at leaft the Relation cannot be expreffed in Numbers.] Such a leffer Quantity, to diftinguifh it from an *Aliquot* Part, is called an *Aliquant* Part. For Example, a leffer Quantity equal to 2 of 3 Parts of another, is called an Aliquant Part thereof.

I have hitherto fpoken of Quantities and their Parts in general; but what is faid is applicable both to what is called *Continued* Quantity, (as Length, Weight, Time, &c.) or to pure *Number*. For every leffer Number is a part of a greater; and is either an *Aliquot* or *Aliquant* Part, becaufe Unity is the common Element or *Aliquot* Part of all Numbers; fo that every Number is a Number of fuch *Aliquot* Parts (as Units are) of every other Number. But this difference is very remarkable, That the *Aliquot* Part of a continued Quantity, confidered properly by itfelf in the nature of a continued Quantity,

is

is only one fingle individual thing, or an Unit; whereas in pure Numbers, one *Aliquot* Part may be a Number greater than Unity. *Ex.* The 3^d part of 12 is 4 (for 3 times 4 is 12.) It is true indeed, that conceiving any continued Quantity to be refolved into 12 equal Parts, one 4th part of it is equal to 3 of thefe parts; yet the Whole and Part are here confider'd only as pure Numbers: for to fay 3 is a 4th part of 12, whatever things we fpeak of, it is but a pure Arithmetical Expreffion; whereas in continued Quantities, becaufe the Part of any thing muft be of the fame nature with the Whole, therefore a 4th of any Length (for *Ex.*) muft be one certain Length; which, in fo far as is neceffary to conftitute a 4th part, is not conceived to be any further divided, but to be an entire Length equal to a 4th part of another. But a more remarkable thing is, that any continued Quantity may have any *aliquot* Part, for that we can conceive at pleafure; but pure Numbers cannot: For fome have no other *aliquot* Part but Unity; as 5 and 7; and others have different parts according to their Compofitions: fo 6 has a half and a third part, but not a 4th part or a 5th part. Again, no Number can have fuch a Part as is denominated by a Number greater than itfelf; for Unity is the leaft part of any Number, and is the part denominated by that Number itfelf; fo 3 has not a 4th part in pure Numbers: But if we confider any Number applicately as fignifying a Number of things which are divifible into any conceivable Parts, then any *aliquot* Parts of one, or of any number of thefe things is poffible. For *Ex.* Tho' a 4th part of 3 is impoffible in pure Numbers, it is poffible when the 3 is applied to things divifible into any Number of Parts as 4.; yet here it muft be carefully remarked, that this is not the 4th part of the Number 3, but of a greater Number into which 3 things are refolved: Therefore every fuch Expreffion as a 4th of 3, or 3 4^{ths} of 2, muft be conceived with this qualification, *i. e.* as poffible only in *applicate* Numbers.

The fame alfo is to be underftood, tho' the Denominator is lefs than the Numerator which is confidered as the *Whole*, when this Number has no fuch Part as is expreffed; as in this *Ex.* a 3^d of 5.

We fhall now gather together thefe Definitions; and from them you'll fee the ground of the Diftinction propofed with the Definition of the Terms.

DEFINITIONS.

1. Every leffer Quantity or Number is in a more general Senfe, a Part of every greater (of the fame Kind,) which is called a *Whole* or *Integer* with refpect to the *Part*. But more particularly,

2. An *Aliquot Part* is that which is contained a certain number of times precifely in the Whole; and that number of times is the Name or Denominator of the Part. *Ex.* If any Quantity or Number is refolved into 3 equal Parts, one of them is an *aliquot* Part, called a 3^d part; fo 2 is a 3^d part of 6. And from the nature of Numbers we have this *Corollary*, viz.

COROLLARY. One is an aliquot Part of every Number, and the Denominator is that Number itfelf. So 1 is the 6th part of 6.

3. An *Aliquant Part* is fuch a leffer Quantity or Number which is not an equal Part; but contains a certain Number of fome equal (or *aliquot*) Parts of the Whole. As when any Quantity or Number is not one *aliquot* Part of a greater, yet is equal to 3 4th parts of that greater: So 9 is 3 4th parts of 12; for a 4th of 12 is 3, and 3 times 3 is 9.

4. An *Integral* (or whole Number) is that which reprefents things abfolutely by themfelves, without any comparifon to other things; *i. e.* they are not confidered as Parts of other things; as when we fay in general 4 Things, or particularly 4 Words: And they are called *Integral* or whole Numbers only in diftinction from Parts.

5. A *Fractional Number* (or a *Fraction*) is that of which each Unit reprefents a certain *aliquot* Part of another thing, as the Whole to which this Part relates, called hence the *Relative Integer*. For *Ex.* 1 5th or 3 5th parts; or 7 13th parts of any thing. And becaufe the Denomination of the Part, which is alfo a Number, muft be expreffed, therefore every Fraction confifts of two Members, or requires two Numbers: for there is the

Number

Number of things directly and immediately reprefented (as in the preceding *Examples*, the Numbers 1, 3, 7.) called hence the *Numerator* of the Fraction; and the Number of equal Parts of which the Relative Integer is fuppofed to be compofed, called the *Denominator*, or the Name of the Part, expreffing the Value of each Unit of the Numerator with refpect to the Quantity of the Relative Integer. So in this *Ex.* 3 5th parts, 3 is the Numerator, and 5 the Denominator; the common way of placing them being to fet the Numerator above the other thus, $\frac{1}{5}$ or $\frac{3}{5}$ Numerator. Denominator.

SCHOLIUM. From this Definition of a Fraction it is plain that the Numerator may either be lefs or greater than the Denominator, or equal to it; for we may as reafonably fay $\frac{7}{5}$ (or 7 5th parts) as $\frac{4}{5}$; if we underftand it according to the Definition, *i.e.* as expreffing 7 things each of which is equal to a 5th of another thing; and not as if 7 were fuppofed to be taken out of 5, which is impoffible. By comparing the Numerator and Denominator, we have this Confequence; *viz.*

COROLLARY. If the *Numerator* is greater, equal to, or leffer than the Denominator, the Quantity expreffed by that fractional Number is greater, equal to, or leffer than the Integer; becaufe the Denominator reprefents all the Parts of the Integer, and the Numerator fhews how many are taken.

And this gives rife to a Diftinction of *Proper* and *Improper Fractions*, as the Numerator is lefs or not lefs than the Denominator. But a more particular Explication of this, with other Diftinctions of Fractions, we muft refer to another place. And here you may *Obferve*, That inftead of the Names *Integral* and *Fractional*, we might as properly call them *Abfolute* and *Relative* Numbers; which do very well exprefs their different Natures: For the firft confiders things fimply and abfolutely in themfelves; and the other confiders things relatively, as Parts of other things.

It is to be alfo *Obferved*, That Fractions are a more general Kind of *applicate* Numbers: For the Numerator (or the Number of things directly defigned) is reftrained; fo that it does not reprefent a Number of any things indifferently; but is limited to a certain Relation to fome other thing: nor does it exprefs any Part of that other thing; but fuch a Part or Parts as the Denominator expreffes: yet while there is no particular thing named as the relative Whole, it is in this refpect a general and *abftract* Fraction, (but not a pure abfolute Number;) fo $\frac{2}{3}$ is a general and *abftract* Fraction; but $\frac{2}{3}$ of a Day is *applicate*. Wherefore in every *applicate* Fraction there are two Denominations to be confidered, which we may call the *Relative* and the *Abfolute*: The firft is the *Denominator* of the Fraction, and the other is the Name of the Integer. But if the Integer is not one, but a Multitude of Things (as $\frac{2}{3}$ of 6 Pounds) that is to be conceived as an Integer or *one* of its own Kind; or rather we are to conceive this Expreffion as a mixt Form reducible to a fimple, wherein the Integer is an Unit of a particular Name; fo $\frac{2}{3}$ of 6 is equal to $\frac{12}{3}$ of 1. But this muft be left to its own place.

GENERAL SCHOLIUM.

Of the different Senfes in which Unity *is Divifible and Indivifible, and the Converfion of Numbers from* Integral *to* Fractional, *and contrarily; fhewing in general wherein their Operations muft be the fame, and wherein they differ.*

Here now is the place to explain the different Senfes in which Unity is divifible or indivifible. And in the firft place this is plain, That *Unity* in its own Nature as Number is *indivifible*; for there can be no Number of Things conceived lefs than *One*: but if we confider the Subject or Thing to which the Idea of *Unity* is applied; as that is capable of divifion into real or imaginary Parts; or as it is really a Collection of diftinct Things united by the Imagination; fo, what is the Subject of *Unity* in one View and under one Denomination, may be the Subject of Multitude under another. For *Ex.* 1 Pound is the fame as 20 Shillings; wherefore it is not the Number *One* that is divifible, but fome

continued Quantity, as a Yard, a Day, &c. or some Number of Things comprehended under a singular Denomination. Hence again we learn to correct a Vulgar Error, viz. That a Proper Fraction (i. e. whose Numerator is less than its Denominator) is a Number less than *Unity*. It does indeed represent a Quantity less than the relative Integer or Unit; but is not a Number less than *Unity*: For the *Numerator*, which cannot be less than 1, is as properly a Number as if it were applied to things under an absolute Denomination: so $\frac{3}{5}$ of a Pound does as truly express three things, as 3 Pound does; differing only in the Value and Way of denominating the things. Again, because one Quantity or Number cannot be referred to another as a Part or Parts, unless that other be really or conceivably divided into, or composed of such a Number of Parts; therefore, strictly speaking, the relative Integer of every Fraction is what we may call a *collective Unit*, or a real Multitude united together in one Whole under a particular Denomination. Hence a Proper Fraction is in effect some lesser Number compared to a Number greater than it, and always greater than Unity. For *Ex.* $\frac{3}{5}$ is 3 things taken out of 5; or 3 5th Parts of a Whole composed of 5 Parts: and in this View only a Proper Fraction is a lesser Number than its relative Integer; yet not as this is *Unity*, but as it is really a Multitude. To have done; The proper Arithmetical Value of One or any other Number is invariable, i. e. One, Two, &c. is always the same Number, in whatever things and however denominated; but take the Number with the Application complexly, there may be a difference; so that what is equal to *Unity*, or any other Number, in one denomination, may be a greater Number applied to another; as 1 Shilling is equal to 3 Groats, or to 12 Pence. The mixt Value is the same in all these, yet the Numbers and Denominations differ: Also what is an Integral Number applied to Things under a proper and absolute Denomination, may be converted to a Fraction or relative Number by applying a relative Name or Denomination; so 3 Shillings is the same as $\frac{3}{20}$ Parts of a Pound. In both Expressions the Number and mixt Value is the same, only the Things are differently denominated in the Application; this being indeed all the difference betwixt Numbers *Integral* and *Fractional*; yet this difference is the occasion that these two Kinds must be handled separately: for the Denominator of a Fraction being also a Number, respect must be had to that in every Operation, which occasions more Work than in Integrals. But still the fundamental Operations of Numbers are those performed by Integral or Absolute Numbers; for the Numerator and Denominator of a Fraction, taken by themselves, are of the same general Nature with every other absolute Number, and can have no other Operation applied to them; and the way of making that Application so as to fulfil all that both the Denominations, Relative and Absolute, (where both are considered) do demand, is the only new thing in the Operations of Fractions. Therefore after the Operations inWhole (or Absolute) Numbers are explained, which will employ the remaining Chapters of this Book; the same shall be done for Fractions in the second Book.

CHAP. III.

ADDITION *of Whole and Abstract* NUMBERS.

DEFINITION.

ADDITION *is the finding one Number equal to two or more Numbers taken all together*; that is, finding the most simple Expression of a Number, (according to the established Notation) containing as many Units as are in all the given Numbers taken together; which is hence called their *Sum*. For *Example*, the *Sum* of these Numbers 8, 17, 24, 675, is found by the following Rule to be 724.

SCHO-

SCHOLIUMS.

1. Before we enter upon the particular Rules of *Addition*, it is neceſſary to make the following Reflection upon the Method of *Notation*, as it is in part the Foundation of all other Rules of Operation; *viz.* That as by any eſtabliſhed Notation, whatever it be, we know how to expreſs any Number, or the whole Series of Numbers in a continued Succeſſion from *Unity*, by the adding or joining Unity after Unity for ever; ſo by Application of this Inſtitution, and the general Axiom, that *The Whole is equal to all its Parts*, we ſee a poſſible way of finding and demonſtrating the Sum of any two Numbers, *viz.* by beginning with one of the given Numbers, (and it is beſt to take the greater) and joining to it all the Units of the other one after another, expreſſing the Sum at every ſtep, according to the Rule of *Notation*, till the laſt Unit is added; and then you have the Sum ſought: becauſe if all the Units in any Number are added ſucceſſively to another Number, the firſt Number (which is nothing elſe but all its Parts together) is certainly added to the ſecond Number. For *Example*, If (according to the preſent *Notation*) it were propoſed to find the

8, 1, 1, 1, 1, 1, 1, 1, ⎫ Sum of 8 and 7: I take 8, and after it I ſet down
9 : 10: 11 : 12 : 13 : 14 : 15. ⎬ all the Units of 7 ſeparately, and by adding them
⎭ one by one to 8, and expreſſing the Sum as it gra-

dually increaſes, the laſt and total Sum is 15. In the ſame manner may we find the Sum of any other two Numbers, or of any Numbers more than two, by firſt adding any two of them, and then to the Sum adding any other of them; and ſo on.

Let us next *obſerve*, What is arbitrary, and what is not ſo in this Operation. In the firſt place, As the whole Syſtem of the Signs of Numbers, both by *Words* and *Figures*, is a pure arbitrary Inſtitution, ſo the Addition or Expreſſion of the Sum of Unity, and any Number, is directly and immediately a Part of that Inſtitution; and therefore has no other Reaſon or Demonſtration but that it is ſo Inſtituted. Again, tho' the Names of all Numbers are contained in the Inſtitution, yet the finding the Sum of any other two Numbers, or calling it by ſuch a Name in the Syſtem, is not immediately a Part of the Inſtitution, but a Conſequence, and therefore Demonſtrable: For the pure and ſimple Inſtitution is all comprehended in the Syſtem of Signs taken in a gradual Succeſſion from Unity, and proceeding by a continual Increaſe of Unity; and therefore contains immediately no Queſtion or Caſe of Addition, but that of adding Unity to Unity, or to any other Number; all other Sums being found by Conſequence from this, which therefore have a proper Demonſtration, different according as the Conſequence is leſs or more remote. As you'll afterwards learn.

Now the Method of Addition moſt immediately connected with the Inſtitution is that above explained: But it is eaſy to perceive how tedious and inſupportable this Method of Addition would be upon any Syſtem of Notation; and as upon different Syſtems, the Remedy of this Difficulty would be leſs or more perfect, ſo the preſent admirable Method of *Notation* affords the moſt eaſy and perfect Rules for *Addition*, (and all other Operations) whereby ſuch Additions are performed by a few and eaſy ſteps, which cannot be done all at once, (as we add Unity to any Number) and would be inſupportably tedious to do by ſo many ſteps as the preceding Method preſcribes: yet this is to be obſerved, that as the eſtabliſhed *Notation* is the Ground-work and Foundation of all, ſo there are ſome ſimple Caſes that can be done no other way; as ſhall be preſently explained,

The moſt ſimple Caſes in any *Problem* are firſt in the Order of Science; and here the Addition of Unity to any Number is the firſt and moſt ſimple Caſe: but as it is contained immediately in the Rule of *Notation*, therefore it is ſuppoſed in the following *Problem*, as a previous and fundamental Principle.

2. This Sign or Character + ſet betwixt two Numbers, ſignifies the Addition of the one to the other; and is a complex or indefinite way of repreſenting the Sum: thus, 3 + 4 ſignifies that 3 and 4 are added together; and we read the Sign by the word *more*. *Exam.* 3 + 4 is 3 *more* 4; and thus it expreſſes the Sum in a complex manner by

the

I

the Parts. And when more Numbers are added, they are joined by the same Sign; thus $3+4+9$ is the Addition or Sum of 3, 4, and 9. The Use of this in particular and determinate Numbers, is chiefly by way of Abbreviation for the neater and shorter Explication of the Work of Addition in particular Examples, as you'll see immediately.

But the principal Use of this Sign of Addition, is for the Expression of the Sum of Numbers, universally or indefinitely represented by Letters in the Algebraick Art. Thus, Any two Numbers being represented by A, B, their Sum is expressed indefinitely thus, $A+B$, and the Sum of A, B and C, is $A+B+C$, and so on; which is the General Rule of the Literal Addition, or Numbers expressed by Letters. Some other particular Considerations relating to this, you'll find in another Place.

Observe again, That as the same Number may be variously represented, either by one simple Expression, or by the simple Expressions of other Numbers variously applied to one another by sundry Operations, whose final Result brings out the same Number; so to express the equality of Value betwixt these different Expressions of the same Number, we use this Sign $=$ set betwixt them; which we use for Abbreviation in explaining the Work of particular Examples in all the common Operations. Thus 8 added to 7 makes 15, which is therefore expressed either complexly $8+7$, or simply 15; and to signify the Equality of these Expressions, we write $8+7=15$, and read it thus, 8 more 7 is equal to 15. Universally, $A+B=D$, signifies that the two Numbers expressed by A and B, are together equal to the Number expressed by D; and so of other Examples of Addition. As $3+4+5=12$, or $A+B+C=D$. Applications of this, in other Operations, you'll find in their Places. I have only this further to say here, That the different Expressions of the same Number, constitute what in the Algebraick Art is called an *Equation*, that is plainly an Equality of Value betwixt two Expressions of Number: In the finding of which, from the Conditions and Circumstances of any Proposition, with the various Changes and Transformations to be made upon them, by the Application of different Operations, whereby one Equation is deduced from another, consists the Algebraick Art of Reasoning; which, so far as the present Undertaking requires, you'll learn as we proceed.

PROBLEM.

To add two or more Numbers into one Sum, or simple Expression.

CASE I. *To add any two Digits or Numbers less than Ten.*

Rule. Take the greater of the two, and to it add all the Units of the other one by one, expressing the Sums gradually according to the Rule of Notation, as explained in *Schol.* 1.

Exam. To add 9 and 6, it is, $9+1+1+1+1+1+1=15$; for adding the 6 Units gradually, and expressing the Sums in the Order of Notation, they make this Series, 9, 10, 11, 12, 13, 14, 15, the last whereof is the Sum sought.

Demon. The Reason of this Rule is already explained; and that there cannot be another Way of adding two Digits, is evident.

SCHOLIUM. Practice by degrees fixes the Sums of all the various Examples of this Case in our Memory; whereby we become capable to pronounce the Answer as readily as the Question is proposed: For upon Reflection it will be found, That we do not always calculate, and add in the manner directed; but know the Sum purely by Memory; which was no doubt acquired by repeated Practice of adding them together Unit by Unit; for it could be done no other way: and therefore it is the only Method we can take to teach young ones, who know nothing but the Names of Numbers in the simple progressive Order from Unity; who may be assisted by their Fingers in this manner, *viz.* Let them tell out the Units of the lesser Number upon their Fingers, then take the greater Number and add to it the Units of the other from their Fingers, expressing the Sums gradually according.

cording.

cording to the progreſſion of Names, and the laſt is the Name of the Number ſought. Again, if thro' any confuſion of Thought, the Sum of two Digits ſhould not readily occur, or if one ſhould pretend to deny or doubt of it, this is the natural and certain way of finding or demonſtrating it; [which may alſo be done by diſſolving one of the Numbers into two Parts, and adding them ſucceſſively to the other Number, which is only reducing the Queſtion to a more ſimple Caſe, where the Sum may be more eaſily remembered.] And this brings to my mind, That in Converſation I have met with Perſons who would affirm that the Addition of two Digits is a thing not properly demonſtrable, but the im-- mediate Effect of an Inſtitution; the contrary of which, I think, I have ſufficiently ſhewn. The Occaſion of this Opinion may be the familiarity we have with theſe ſimple Caſes from our firſt acquaintance with Numbers; ſo that remembring the Sums as readily as we do the Sum of Unity and any Number, we are apt to fancy we came by them without any Reaſoning or Calculation, becauſe we have them ſo now. I muſt alſo obſerve, That the Method of our common Books of Arithmetick may have contributed to this; for in theſe we have but one general Caſe and Rule of Addition, in which it is ſuppoſed that we know already, or can find the Sum of any two Digits; and this perhaps is done upon a Suppoſition of its being ſimple and eaſy: But this would be no reaſon for omitting it in a Work deſigned for a juſt and rational Syſtem; which muſt therefore explain the Connection and Dependence of all the Parts of the Science upon their firſt Principles, and upon one another.

In the laſt place, then, ſince this Caſe is ſuppoſed in all other Caſes of Addition, it is neceſſary the Learner know the Sum of all its Examples as readily as the Queſtion is propoſed. In order to which; I ſhall expreſs them all in the following Table; from which they may be got by heart more eaſily by thoſe who are Beginners in this Science.

TABLE ſhewing the Sums of any two Digits.

1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	10	1
	4	5	6	7	8	9	10	11	2
		6	7	8	9	10	11	12	3
			8	9	10	11	12	13	4
				10	11	12	13	14	5
					12	13	14	15	6
						14	15	16	7
							16	17	8
								18	9

The Conſtruction of the *Table* is obvious, and the Manner of uſing it is this: Take the greater of the two Digits, whoſe Sum is ſought, in the upper Line, and the leſſer on the Right-hand Column; in the ſame Line with this, and under the other ſtands the Sum. So under 8, on the Head and in the ſame Line with 6, on the Side ſtands the Sum 14.

CASE 2. *To add any two or more Numbers into one Sum.*

Rule 1. Place the given Numbers under one another, ſo that the Figures in like places of each be directly in one Column, [*i. e.* all Figures in the firſt or Units place in one Column; all in the ſecond place, or Tens, be alſo in one Column; and ſo on in the Order of Places.] Then *ſecondly*, begin at the Units place of the loweſt Line or Number; add that Digit to the next above of the ſame Place and Column; and to this Sum add the next Digit, and ſo on till all the Digits in that Column are added; then if the Sum is leſs than 10, ſet it down under the Figures added; and add up the ſecond Column the ſame way; and ſo on thro' all the Columns, taking the Figures in each according to their ſimple Value. But if the Sum of the firſt, or any other Column exceeds 10, then it is either a preciſe Number of Tens (as 20, 30, &c.) or it is ſuch a Number of Tens with ſome

Number.

Number lefs than 10, (as $24 = 20 + 4$; or $68 = 60 + 8$.) In the firft Cafe write down 0, and in the other the Number over a precife Number of Tens, (as 4, if it is 34; or 8, if it is 68) and for every 10 in the Sum carry 1 to the next Column; *i. e.* add the Number of Tens in the Sum of every Column to the next Column. Having thus gone thro' all the Columns, the Number of Tens in the Sum of the laft Column is to be fet down on the left of all the Figures already found; or, the whole Sum of the laft Column is fet on the left of all the preceding Figures : and all thefe Figures thus found and placed, exprefs the Sum fought.

$$
\begin{array}{r}
654 \\
243 \\
\hline
897 \ \text{Sum.}
\end{array}
$$

Ex. 1. $654 + 243 = 897$. wrought as in the Margin thus : Beginning at the Units place of the lower Number, I fay, $3 + 4 = 7$, which I fet under the Numbers added; then $4 + 5 = 9$; and laftly, $2 + 6 = 8$; and the total Sum is 897.

$$
\begin{array}{r}
897 \\
968 \\
\hline
1865 \ \text{Sum.}
\end{array}
$$

Ex. 2. $897 + 968 = 1865$. Thus, $8 + 7 = 15 = 10 + 5$; therefore I fet down 5, and carry 1 to the next place; then $1 + 6 = 7$, and $7 + 9 = 16 = 10 + 6$, for which I fet down 6, and carry 1 to the next place; then $1 + 9 = 10$, and $10 + 8 = 18$, which being fet down, the total Sum is 1865.

Let the Learner practife more Examples of this Kind to himfelf.

SCHOLIUMS.

1. If there are more than two Numbers to be added, this Rule plainly fup-pofes, that we can readily, in our Mind, add any Digit to any other Number; which therefore might have been confidered as a particular Cafe : But, rather than make too many Cafes, I have made this Suppofition; nor has it any thing contrary to good Order, becaufe it is indeed no other than a particular Cafe comprehended under the general Rule here delivered : fo that by practifing Examples of this fimple Cafe, we foon acquire the Capacity fuppofed for more complex Cafes. For, if the greater Number has, in the place of Units, fuch a Digit, as added to the other Digit, makes a Sum lefs than 10, the Total is plain; thus, $64 + 2 = 66$; $22 + 7 = 29$. But if the Sum of the two Digits exceeds 10, what's over belongs to the place of Units of the Sum, and 1 for the 10 is to be added to the remaining part of the greater Number. Thus $46 + 8 = 54$, *viz.* $6 + 8 = 14$; which makes 4 in the place of Units of the total Sum, and then $1 + 4 = 5$ in the place of Tens. Alfo $397 + 9 = 406$ for $7 + 9 = 16$, then $1 + 39 = 40$.

There is alfo another Suppofition in this Rule, *viz.* That we can readily perceive how many Tens are contained in any Number, or in the Sum of any Column of Digits, and alfo what remains over. Now, tho' thefe do indeed belong to other Rules, yet the Rule of *Notation* has already taught us them, in the Cafes here fuppofed : For, if we write down any Number, the Figure in the place of Units is the Number over all the Tens contained in it, and the remaining Figures on the Left-hand, taken by themfelves, exprefs the Number of Tens; as has been explain'd in the Rule of *Notation* (fee *Corol.* 4. after the Problem in Chap. 2.) thus, 40 is 4 Tens and 0 over, 87 is 8 Tens and 7 over, 124 is 12 Tens and 4 over. In fmall Examples the Number of Tens is eafily perceived without writing down the Sum, and in greater Numbers write it down. But there are other Methods of managing Addition, whereby the Number of Tens will be mark'd out in the Operation, which fhall be prefently explained; but I fhall firft make fome larger Examples, and fhew the Application of the preceding Rule to them.

Ex.3d.	Ex.4th.	Ex.5th.
868	87869	34
753	26100	596
209	7895	789
546	3746	6897
370	65432	2540
921	789576	4698
689	456789	246
796	324160	86532
5152	1761567	78063
		65
		180460

The Operation of *Ex.* 3. is thus, $6+9=15$; $+1=16$; $+6=22$; $+9=31$; $+3=34$; $+8=42$; which is 2 for the first place and 4 to carry to the 2d: thus, $4+9=13$; $+8=21$; $+2=23$; $+7=30$; $+4=34$; $+5=39$; $+6=45$; which is 5 for the 2d place, and 4 to carry to the 3d: thus, $4+7=11$; $+6=17$; $+9=26$; $+3=29$; $+5=34$; $+2=36$; $+7=43$; $+8=51$; which being the Sum of the last Column, is all written down; so the total Sum is 5152.——Observe that you are to read the Operation thus, $6+9=15$, then $15+1=16$, &c. But to prevent writing the last Sum twice, I have separated it from the Number to be next added to it by a Semicolon, by which you must understand, that it is added to the next Number. [But not that these Expressions are all equal, viz. that $6+9$, $15+1$, &c. are equal.]

After the same manner examine the other two Examples, for your own Practice.

2. If there are many Numbers to be added, so that the Sum of every Column be a great Number, we may save the Memory being too much burdened, and the Work render'd thereby more difficult and uncertain, by various means; as,

First, By making a Point at every 40 or 50, or 100. Thus, Add up the Column till you have a Sum containing any Number of Tens you please, as 50 or 100, making a Point at the last Figure which makes up your Number of Tens, and if there is any excess over Tens, add it to the next Figure; and so go on thro' all the Column: When you have finished it, if the particular Sum after adding the last Figure is less than 10, set it down in the place of the total Sum (or Answer,) and for every Point carry as many Units to the next place as there are Tens in the Number pointed, (i. e. 3, 4, 5, or 10, for 30, 40, 50, or 100,) and if the last particular Sum exceed a Number of Tens, set down the Excess, and join that Number of Tens to the Number arising from the Points, and carry the Sum to the next Column; and thus go thro' them all.

So in the annex'd *Example*, which I point at 50, I proceed thus; $9+7$ is 16, and 8 is 24, and 7 is 31, and 9 is 40, and 8 is 48, and 9 is 57; here I point at the 9, and take the 7 which is over 50, and say, $7+6$ is 13, and 9 is 22, and 4 is 26, and 5 is 31, and 6 is 37, and 7 is 44, and 5 is 49, and 7 is 56, which makes a Point at the 7, and 6 over; with which I proceed, saying, $6+9$ is 15, and 8 is 23; therefore I write down 3 in the Sum, and I have 2 (for the 20) to be added to the Number of the two Points, viz. to 10 (for every Point is here 5) and so I have 12 to carry in to the next Column; which I add in the same manner.

To prevent blotting of Accounts, which is the Objection made against this Practice, you may make the Points upon a Shred of Paper applied to the Column.

There is also another useful Method of summing up long Columns, viz. by distributing the Numbers into Parcels of 9 or 10 Numbers in each, (so that the Sum of each Column in every Parcel shall never exceed 100) adding each Part by itself, and then adding their Sums into one total Sum. As in the following *Example*, which needs no further Explanation.——

528
269
587
675
367
986
895
764
489
76
29
38
19
27
38
27
9
5803

E

D E-

DEMONSTRATION *of the Rule of* CASE 2:

678
957
834
928
653
356
890
348
269 . . 5913
372
908
735
637
943
259
167
896
725 . . 5702
Total 11615

This we deduce from the Nature of Notation, and the common Axiom, *That the Whole is equal to all its Parts : Thus ;* every Number confifts of as many Parts as there are fignificant Figures in it; and the Figures ftanding in the fame Places of different Numbers are the fimilar or like Parts of thefe Numbers; which being therefore added together according to their fimple Values, the Sum has all the fame fecondary or local Value, *viz.* That of the Place in which thefe Figures do all ftand: But (by *Cor.* 4. *Chap.* 2.) any Number is equal to as many Tens as the Figures above the place of Units taken by themfelves do exprefs, and to that Number of Units more; (fo 248 is 24 Tens + 8.) And again, 1, or any Digit ftanding in any place is equal to 10 times the Value of the fame Digit in the preceding place : wherefore the Sum of the Digits ftanding all in the like places of different Numbers, is equal to as many Units of the Value of the next higher place as there are Tens in the Sum, and to as many Units of the place added as are over that Number of Tens. Hence we fee the Reafon of all the Parts of the Rule : for the Parts of Numbers added muft be fimilar, elfe the Sum is falfe; fo 4 + 7 = 11, which fuppofes them to be both of one local Value; but if the 4 be in the place above the 7, the Sum is not 11, but 47 : This explains the firft Part of the Rule. Then, by carrying forward the Number of Tens found in the Sum of every Column (added according to the fimple Values) and adding that Number to the next Column (with which it has a fimilar Value) we actually add all the Parts of each of the Numbers given ; and do add them fo, as that all the Units of each particular Digit fet down in the total Sum have one local Value, which is the fame that the Parts to whofe Sum it belongs have in the Numbers added; but the Parts are equal to the Whole, therefore the Numbers fet down according to the Rule, exprefs the true Sum.

As to the fecond Part of the Rule, *viz.* The carrying forwards the Number of Tens, we may fee the Reafon of it another way; Thus, fuppofe every Column added by itfelf, and the Sums fet down feparately fo as the firft Figure of the Sum ftand in the place of that Column; and let all thefe Sums be again added in the fame manner; and fo on till we have the moft fimple Expreffion of the Total of the given Numbers. Now this is a very natural way of Operating, which carries its Reafon with it; but it is alfo plain, that the carrying forward the Tens has the very fame Effect; for it is only doing that all at once, which is here done at feveral fteps, and is thereby a more compendious Way of finding the Number fought, and therefore preferable.

Example I.

348
695
476
587
290

Sum of Units	26
Sum of Tens	370
Sum of Hundreds	2000
Total Sum	2396

Example II.

$$5789$$
$$6958$$
$$4746$$
$$8984$$
$$4897$$
$$6518$$
$$7968$$
$$9376$$

56	Sum of Units.
480	— of Tens.
5700	— of Hundreds.
49000	— of Thousands.
06	Sum of Units.
130	— of Tens.
1100	— of Hundreds.
14000	— of Thousands.
40000	— of Ten Thousands.
55236	Total.

§. 2. Of the Proof of Operations in *Arithmetick*, and particularly *of* ADDITION.

THE Proof of any Operation in Arithmetick is some other Operation, by which we are made more certain that the first Work is right performed, and the true Answer found.

It is different from the Demonstration of the Rule; for this shews that if the Rule is followed, the true Answer will be found: But the Proof supposes the Truth of the Rule, and only shews whether we have observed it in doing the Work.

The Proof of any Operation ought naturally to be another easier than itself, in which there is less hazard of erring; otherwise we are not more certain of the one than of the other: And if we be very scrupulous, we may require again a Proof of this second Work, by a third easier than the last; and so on till we have the most simple Operation. But a little practice puts us beyond the need of all this; and we may be satisfied with any one Proof, even tho' it is not an easier Work than that to be proved, because there is less hazard of Erring in both than in one. However, the easiest Proofs, if they are not tedious, are always preferable.

PROOF of ADDITION.

If we should propose a more simple Kind of Operation for Proof *of Addition,* we have none but the adding Unit by Unit, already spoken of in *Notation.* But the simplicity of this Method is infinitely overbalanced by its tediousness; therefore we must be content with another Application of the same Rule; which may be made different ways.

1. If the Numbers have been added all together without distributing them into Parcels, (as has been explained) then by making such a Distribution, and adding them that way, we shall prove the Answer which was found the other way; for the total Sum must be the same in both Methods: But if it has been at first wrought this way, we may make a Proof by making a different Distribution.

2. Whatever Way the Work is done at first, we may do it again the same way, only beginning at the upper Line, and adding downwards.

E 2   3. There

3. There is another way ingenious enough, and very easy; but as it suppoſes a ſmall Capacity in Subtraction, it may ſeem not agreeable to good Order to propoſe it in this place: For tho' one Operation or Rule may no doubt be made ſerviceable to another, yet every thing ought to be in its Place; therefore you'll ſay, that according to the utmoſt ſtrictneſs of Method, what requires Subtraction, ought to come after Subtraction. But the thing ſuppoſed is no more than that we can readily know the difference betwixt 9 and any Number leſs than 18; and this Capacity cannot but be already acquired by the practice of Addition, (of which Subtraction is but the Reverſe.) There is alſo ſomething in it which produces the ſame Effect as *Diviſion;* but as it does not require the Rule of Diviſion, that furniſhes no Objection: Therefore I ſhall explain this Method here as its proper place.

Rule. Take each of the given Numbers ſeparately, and add all their Figures together as ſimple Units; and in doing ſo, when you have made a Sum equal to 9, or greater than 9, but leſs than 18, neglect the 9, taking what's over and add to the next Figure; and go on ſo till you have gone thro' them all, and mark what is over or under 9 at the laſt Figure; but if the Sum of all the Figures is leſs than 9, mark that Sum. Do the ſame with each of the given Numbers, ſetting all theſe Exceſſes of 9 together in a Line, (in any Order,) then ſum them up the ſame way, marking the exceſs of 9 as before, (or what the Sum is leſs than 9.) Laſtly, do the ſame with the total Sum, and what is under 9, or over any Number of 9's in this, muſt be equal to the Exceſs (or Number leſs than 9) laſt marked; elſe the Work has not been right performed.

I ſhall explain this Practice by one Example.

Example.

$$
\begin{array}{r}
2743 \\
4678 \\
5265 \\
\hline
12786
\end{array}
\left|\;
\begin{array}{c}
\text{exceſs of 9's} \\
7 \\
7 \\
1 \\
\hline
6
\end{array}
\right\}
$$

Beginning with the upper Line, I work thus; $2+7=9$; then $4+3$ $=7$, which I have ſet down on the Right. Then to the ſecond Line, $4+6=10$, which is 1 over 9; then $1+7+8=16$, which is 7 over 9; which I ſet under the Figure laſt found. Again, to the third Line, $5+3+6=14$, which is 5 over 9; then $5+5=10$, which is 1 over 9. Then I do the ſame with the Line of the Figures now found, *viz.* $1+7+7=15$, which is 6 over 9; and finding the ſame Exceſs of 9's in the Sum (12786,) I conclude the Work is right performed.

Obſerve, It will have the ſame Effect if, inſtead of ſetting down the Exceſs of 9's in the ſeveral given Numbers, we carry the Exceſs of one Line into another, and only mark the laſt Exceſs; which ought to be the Exceſs in the total Sum.

Demonſtration. In order to ſhew the Truth of this *Rule,* I muſt premiſe and demonſtrate this *Lemma,* which will be uſeful alſo afterwards.

L E M M A.

The Figure that ſtands in any place of a Number, taken in its ſimple Value, is equal to what will remain after 9 is taken out of the compleat Value as oft as poſſible; i. e. after all the 9's contained in it are taken away.

For *Example,* If all the 9's contained in 700 are taken away, there remains the ſimple Number 7.

Demon. Any Figure ſtanding in any place of a Number is equal to ten-times the Value of the ſame Figure in the next lower place, (by what has been ſhewn in Notation;) *i. e.* equal to 9 times $+1$ time that Value, (becauſe $9+1=10$.) But 9 times any Number is a preciſe Number of 9's; which being taken away, there remains once the Value of it in that next place, and this again is equal to 9 times $+1$ time the Value of the ſame Figure in the next lower place, and the 9 times being taken away, the 1 time remains; and ſo on till you bring it down to the place of Tens, where it is equal to 9 times its ſimple Value $+$ once that Value; and the 9 times taken away, there remains the ſimple Value:

But

But thus we have fuupofed all the 9's to be taken out of it, and confequently the *Lemma* is true.

Corollary. The Sum of all the Figures in any Number, taken as fimple Units, is the Remainder after as many 9's are taken out of that Number, as are to be found feparately in the compleat Value of each of the faid Figures (becaufe each of thefe Figures taken fimply, is the Excefs of the 9's contained in that Part,) and if that Sum is lefs than 9, it is the Remainder after the 9's contained in the given Number are taken away: but if it is not lefs than 9, the Remainder, after all the 9's are taken out of it, is the Remainder of 9's in the given Number: For it is plain that there can be no more 9's in any Number than what are in the feveral Parts and in the Sum of the Excefs of 9 in the fame Parts.

Now from this *Lemma* and *Corollary*, the Demonftration propofed will be plain: For by adding the Figures of any Number according to the Rule, it is evident we find the Excefs over all the 9's contained in their Sum (taken as fimple Units.) And this is the Excefs of all the 9's contained in the faid Number by the *Corollary*. But again, the Excefs of 9's in each of two or more Numbers being taken feparately, and the Excefs of 9's taken alfo out of the Sum of the former Excefles, it is plain this laft Excefs muft be equal to the Excefs of all the 9's contained in the Total of all thefe Numbers, (the Parts being equal to the Whole.)

<div align="center">SCHOLIUMS.</div>

1. In the Demonftration of the preceding *Lemma*, I have taken it for a Truth, that 9 times any Number is a precife Number of 9's; *i. e.* that it is equal to that Number of times 9, (without any thing over.) For *Example*, that 9 times 7 is 7 times 9; for the Demonftration of which (if it is required) I refer you to Chap. 5. as I have done already in a like Cafe.

2. To this Proof it is commonly objected, That a wrong Operation may appear to be true; which muft be owned: for if we change the Places of any two fignificant Figures in the Sum, it will ftill appear right. So in the preceding Example, the true Sum is 12786: But fuppofe, thro' miftake, it had been 12768; it is plain this Method of Proof would make it appear right, becaufe there is the fame Excefs of 9's where there are the fame Figures, whatever order they ftand in. But then confider, a true Sum will always appear true by this Proof, (for that is demonftrated) and to make a falfe Sum appear true, there muft be at leaft two Errors, and thefe oppofite to one another; *i. e.* one Figure greater than it ought to be, and another as much lefs; and if there are more than two Errors, they muft always balance among themfelves; *i. e.* the Sum of the Figures that are greater than they ought to be, muft always be equal to the Sum of the Figures that are deficient; elfe it is plain, a falfe Sum will not appear to be right. But now if we confider what an exceeding great Chance there is againft this particular Circumftance of the Errors, and how fimple the Proof-work itfelf is, we may truft to this Proof as fafely as to any other.

<hr>

<div align="center">

CHAP. IV.

SUBTRACTION *of Abstract Whole Numbers.*

DEFINITION.

</div>

SUBTRACTION *is the taking one Number out of another; or finding the* Difference *betwixt two Numbers; i. e.* the moft fimple Expreffion of that Number whereby the greater of two given Numbers exceeds; or the leffer comes fhort of the other.　　　　　　　　　　　　　　　　　　　　　　　　　　　　*Example.*

Example. The Difference betwixt 8 and 3 is 5: Betwixt 48 and 19 is 29.

Observe also, That for distinction the greater Number is called the *Subtrahend,* and the lesser the *Subtractor;* and the Number sought, or the Difference, is called also the *Remainder, i. e.* what remains after the lesser is taken out of the greater.

SCHOLIUMS.

1. As the Effect of *Subtraction* is plainly the Reverse of *Addition*; so is its Operation: Wherefore we have this first to observe, That by a continual retracting Unity after Unity (by the reverse of what was done in Addition) the difference of any two Numbers may be found; which shews an immediate Connection betwixt this Work and the Rule of Notation. But the insufferable tedioufness of this Method is removed by the following Rule, whereby that is done by a few easy steps, which cannot be done all at once, and ought not to be done by more steps than are necessary: and this we owe to the Method of Notation. But we must also observe, That as the subtracting Unity from any Number is the most simple Case, so it is immediately contained in the Rule of Notation, and presupposed in the following *Problem*; and is also the only Method by which one Digit can be subtracted from another; which therefore I make the first *Case* of the following *Problem,* as that upon which all other *Cases* depend.

2. This Sign or Character — set betwixt two Numbers, signifies the *Subtraction* of the one from the other; and is a complex or indefinite way of representing the *Difference.* Thus 7 — 4 signifies that 4 is taken from 7; and we read the *Sign* by the word *less. Examp.* 7 — 4 is read, 7 less 4: and thus it expresses the difference in a complex Manner, by the Whole and Part taken away. And if more Numbers are successively *subtracted,* we prefix the same Sign to each of them; thus, 12 — 4 — 3 signifies that 4 is taken from 12, and then 3 from the first Remainder; or, which is the same thing, that 4 and 3 both (*i. e.* 7) is taken from 12. By this *Sign* we explain in a neat and brief way the Work of particular Examples in *Subtraction*; as you'll presently see.

But the principal Use of this Sign of *Subtraction* is for the Expression of the difference of Numbers represented by Letters in the *Algebraick Art.* Thus, A — B expresses the difference of A and B; A — B — C, the difference of A and B + C; for by taking all the Parts of one Number successively from another, we take the Whole of that Number from this. And this is the *General Rule of the Literal Subtraction*: other particular Cases you'll earn in another place.

PROBLEM.

To Subtract *one Number from another; or, to find their* Difference.

CASE 1. *To find the Difference betwixt two Digits.*

Rule. This is to be done by a continual retracting each *Unit* of the *Subtractor* successively from the *Subtrahend,* and expressing the Differences gradually, according to the *Rule of Notation.*

Example. 9 — 4 = 5. For 9 — 1 — 1 — 1 — 1 = 5. The gradual Series of Differences being 8, 7, 6, 5; because 9 — 1 = 8, — 1 = 7, — 1 = 6, — 1 = 5; the Number sought.

SCHOLIUM. As all other Cases necessarily suppose this one; the Answers of all its Examples ought to be ready in the Memory: and therefore I shall put them all in the following *Table,* for their Use who are mere Novices in *Arithmetick.* But I must here *observe,* That as this is only the *Reverse* of *Addition,* whoever is Master of that, (as they ought to be before they enter on *Subtraction*) will be able at once to pronounce the difference of any two Digits.

TABLE for finding the Differences *of any two Digits.*

1	2	3	4	5	6	7	8	9	0
0	1	2	3	4	5	6	7	8	1
	0	1	2	3	4	5	6	7	2
		0	1	2	3	4	5	6	3
			0	1	2	3	4	5	4
				0	1	2	3	4	5
					0	1	2	3	6
						0	1	2	7
							0	1	8
								0	9

The Use of this Table will, I think, be obvious: Seek the *Subtractor* on the Right-side Column, and against it under the *Subtrahend,* found in the Upper-Line, is the difference sought. *Examp.* Under 7 on the Head, and against 4 on the Side, you find 3, the difference of 7 and 4. Or also thus, Take the *Subtrahend* on the Head, and in the Column under it, seek the *Subtractor,* against which on the Side is the Difference.

CASE 2. *To* Subtract *any Number from another.*

Rule 1. Write the one under the other according to the Order of Places, (and most commonly the *Subtrahend* is set above the other, tho' this is not necessary.) Then, Secondly, take the difference betwixt each Figure of the *Subtractor* and its Correspondent in the like place of the *Subtrahend,* beginning at the place of Units, and set the Remainders under them in order; but if the Figure in any place of the *Subtrahend* is less than its Correspondent in the *Subtractor,* add 10 to that Figure, and subtract from the Sum, and set down the Digit which remains; then add 1 to the next Figure of the *Subtractor,* and take the Sum from its Correspondent in the *Subtrahend,* and go on so, adding in the same manner whenever the Figure of the *Subtrahend* is least. All the Figures written down, express the difference sought.

Subtrahend 876
Subtractor 524
Difference 352

Examp. 1. The difference of 876 and 524 is 352, as in the Margin. The Operation is thus, $6 - 4 = 2$, which I set down; then $7 - 2 = 5$, and $8 - 5 = 3$; and the difference sought is 352.

Subt^d. 74625
Sub^x. 47382
Diff. 27243

Examp. 2. The difference of 74625 and 47382 is 27243. Whose Operation is thus, $5 - 2 = 3$; then $2 - 8$ is impossible, therefore I take 12, and say, $12 - 8 = 4$; then (because 10 was added to the last place of the Subtrahend) I add 1 now to the Subtractor, and say, $1 + 3 = 4$; then $6 - 4 = 2$. Again, $4 - 7$ cannot be; therefore I take 14, and say, $14 - 7 = 7$. Again $1 + 4 = 5$, and $7 - 5 = 2$. And so the compleat difference is 27243.

You may examine these few more Examples in the same manner.

Examp. 3.	*Examp.* 4.	*Examp.* 5.	*Examp.* 6.
87234	2835	345	58034
53890	925	79	928
33344	1910	266	57106

SCHO-

SCHOLIUM. This Rule suppofes we can readily in our Mind fubtract any Digit from the Sum of any leffer Digit and 10; which may be eafily admitted: but to make the Difficulty lefs, (if there can be any) we may do the Work thus; Subtract from the 10, and add the Remainder to the Figure of the *Subtrahend* to which the 10 fhould have been added, and fet this Sum down.   So in *Examp.* 5. fay 9 from 5 cannot, but from 10, and 1 remains; then $1+5=6$; which is fet down.

D E M O N S T R A T I O N *of the Second Cafe.*

1. Where all the Figures of the *Subtractor* are lefs than their Correfpondents in the *Subtrahend,* the difference of the Figures in the feveral like places fet in the fame place, muft all together make the true difference fought; becaufe as the Parts make up the Whole, fo muft the differences of all the fimilar Parts of any two Numbers make the total Difference of the Wholes, of which thefe are the fimilar Parts.

2. Where any of the Figures of the *Subtrahend* is lefs than the Correfpondent in the *Subtractor,* the 10 which is added by the Rule you are to fuppofe to be the Value of an Unit taken from the next higher place, (which by *Notation* is equal to 10 in this Place,) and then the 1 added to the next place of the *Subtractor* is to diminifh the next place of the *Subtrahend* by 1 more than is contained in the *Subtractor,* becaufe that 1 was fuppofed to be already borrowed from it, and applied to the preceding Place according to its Value there: fo that inftead of adding 1 to the *Subtractor,* we may take 1 from the *Subtrahend*; *i. e.* take the *Subtractor* from the *Subtrahend* Figure leffened by 1. But the Effect is the fame, fince either way that 1 is taken from the *Subtrahend*; as it ought to be, fince it is already applied to the laft place; which is only taking from one Part, and adding as much to another, whereby the Total is never changed: And by this means the *Subtrahend* is refolved into fuch Parts as are each greater than (or equal to) the fimilar Parts of the *Subtractor.*   So in order to fubtract 26 from 52,
52   refolve the 52 into thefe Parts, 40 and 12, fo that the 12 correfpond to the 6 of
26   the *Subtractor,* and the 40 to the 20; which is in effect done by the Rule: for we
—   fay, 6 from 2 cannot, but from 12, and 6 remains; then it is plain, that for 5 we have but 4 in the next place, and it is the fame thing to fay 2 from 4, or 3 from 5.

But there is another thing to be accounted for, which the Rule fuppofes, *viz.* That the Difference betwixt any Digit and the Sum of any leffer Digit and 10, will always be lefs than 10; the Truth of which is plain; for fince that leffer Digit wants at leaft 1 of the greater, (to make a Part equal to the Subtractor) that 1 being taken from the 10 added, there cannot remain above 9.

SCHOLIUM. If it be propofed to fubtract two or more Numbers from any one, or one from more; or laftly, more than one from more than one; the beft and moft fimple way is, firft to add thefe more together, and then fubftract: So let it be propofed to fubtract 560 from $467+235$, as in Example 1. below; or, $345+432$ from 978, as in Example 2. or, $3072+5678$ from $2578+9631$, as in Example 3.   Alfo when more Numbers are to be fubtracted out of one, it may be done by taking away firft one of them, and then out of the Remainder take another, and fo on till they are all fubtracted; and this is called *Continual Subtraction.*

	Example 1.		*Example 2.*			*Example 3.*

	Example 1.
	467
	235
Sub^d.	702
Sub^r.	560
Diff.	142

	Example 2.
Sub^d.	978
	345
	432
Sub^r.	777
Diff.	201

	Example 3.
	2578
	9631
Sub^d.	12209
	3072
	5678
Sub^r.	8750
Diff.	3459

The PROOF *of* Subtraction *may be made either by* Addition *or* Subtraction.

1. By *Addition*, thus; Let the Remainder be added to the *Subtractor*, and the Sum ought to be equal to the *Subtrahend*: For this is restoring back what was before taken away.

2. By *Subtraction*, thus; Subtract the Remainder from the *Subtrahend*; and this Remainder ought to be equal to the former *Subtractor*: Because which soever of the two Parts that make up any Whole is taken away, the other remains. I leave you to apply these Proofs in the preceding Examples.

If we have subtracted more Numbers out of one by continual Subtraction, the Proof of this Work will plainly be, *Adding all the Subtractors and the last Remainder* into one Sum, which must be equal to the *Subtrahend*.

CHAP. V.

MULTIPLICATION *of Whole and Abstract* NUMBERS.

DEFINITIONS.

1. MULTIPLICATION *is the taking any Number a certain number of times: or, finding a Number which shall contain any given Number a certain proposed number of times;* i. e. as oft as that other Number contains *Unity.* For *Ex.* If it is proposed to take 48, 7 times; or find that Number which contains 48, 7 times; the Answer by the following Rule, is, 336.

2. The Number to be multiplied is called the MULTIPLICAND; the Number by which it is multiplied, (or the number of times it is to be taken) is called the MULTIPLIER; and the Number found is called the PRODUCT. The *Multiplicand* and *Multiplier* are also called the FACTORS (of the *Multiplication*) without distinction; because they *make* the *Product*, or Number sought.

SCHOLIUMS.

1. I know there may be made a more general Definition of *Multiplication*, comprehending in it also what is called, *Multiplication by Fractions:* But that being really a mixt Operation of *Multiplication* and *Division*, I thought it more reasonable to make the Definition here to agree only to whole Numbers, which is proper and pure *Multiplication*, according to the more common Sense of the Word; and when you learn the other, it is only joining them both together to make that more general Definition: which I shall do in its proper place.

F

2. The

2. The Senfe and Effect of *Multiplication* (of whole Numbers) is the fame with *Addition*; for it's plain, that if we take the *Multiplicand*, and write it down as oft as there are Units in the *Multiplier*, the Sum of all thefe, taken by *Addition*, is the Number fought.
For *Ex.* to multiply 48 by 7, or to take 48, 7 times; I fet down 48, 7 times (as in

48
48
48
48
48
48
48
————
336

the Margin) and find the Sum 336. But *Multiplication* is a Method of finding the fame Number more eafily and expeditioufly. For *Ex.* to multiply any Number by 468; what a tedious and intolerable thing would it be, to write down the Multiplicand 468 times? But by the following Rules this is prevented, and the Number fought is found by an eafy Operation.——*Multiplication* then is only a compendious *Addition*, limited to that particular Cafe wherein all the Numbers to be added are equal to one another, (or the fame Number:) For it's this Circumftance that affords us a more eafy Method of working than by the general Rule of *Addition*. Yet there are fome fimple Cafes which admit of no Compend: Thefe are the Multiplication of Numbers under 10, or the Digits, by one another; which are the *primitive Operations* in *Multiplication*, upon which all other Cafes depend. We muft therefore explain *Multiplication* alfo in two Cafes, as we have done *Addition*. But whereas the *primitive* Cafes of *Addition* depend immediately upon the Rule of *Notation*; the *primitive* Cafes of *Multiplication* depend immediately on *Addition*. For we need not go back to *Notation* for thefe; fince we have, by an intermediate Step in *Addition*, gained an eafier way of doing them.

3. This Sign or Character × fet betwixt two Numbers, fignifies the *Multiplication* of the one by the other (taking either of them for the *Multiplier* or *Multiplicand*; which does not alter the Product, as will be afterwards demonftrated) and is a Complex or indefinite way of reprefenting the Product. Thus 7 × 3 fignifies that 7 is multiplied by 3, or 3 by 7; which we read 7 times 3, or 3 times 7; whereby the Product is expreffed in a complex manner by the *Factors*: And if more Numbers are fucceffively or continually (as it's called) multiplied together, the fame Sign is prefix'd to each fucceffive Factor. Thus 4 × 6 × 3 expreffes the Product made by multiplying 4 by 6, and this Product again by 3; and fo of more Factors, (which will ftill be the fame Product in whatever order the Factors are applied, as will be demonftrated.) By this Sign we explain in a neat and brief way the Multiplication of particular Examples, as you'll prefently fee.

But the principal Ufe of this Sign of Multiplication is in the Algebraic Art, to exprefs the Product of two or more Numbers in Letters. Thus, A × B is the Product of A and B; alfo A × B × C the Product of A, B and C, and fo on. But when two or more Numbers are expreffed, each by one Letter, the Product is alfo expreffed by thefe Letters fet together without the Sign. Thus A B is the Product of A and B, A B C that of A, B, and C. This is the general Rule of the *Literal Multiplication*; other particular Cafes you'll learn afterwards.

PROBLEM.
To multiply any Number by another.

CASE I. *To multiply one Digit by another.*

Rule. Add the *Multiplicand* to itfelf as oft as there are Units in the *Multiplier*, and you have the Product fought.——*Examp.* To multiply 6 by 4, I fay 6 + 6 (= 12) + 6 (= 18) + 6 = 24, the Number fought. But the Products of all the Digits ought to be ready in the Memory; which are eafily got by the help of the following Table.

TABLE of *MULTIPLICATION* for the simple Digits, or Numbers lefs than 10.

1	2	3	4	5	6	7	8	9	1
	4	6	8	10	12	14	16	18	2
		9	12	15	18	21	24	27	3
			16	20	24	28	32	36	4
				25	30	35	40	45	5
					36	42	48	54	6
						49	56	63	7
							64	72	8
								81	9

USE of the TABLE.——Seek the greater of the two Digits in the upper Line, and under it againft the leffer, taken in the right-fide Column, is the Product fought. *Ex.* To multiply 8 by 4, I take 8 in the upper Line, and under it againft 4 I find 32 the Product. Again, to multiply 7 by 9, take 9 in the upper Line, and under it againft 7 on the fide, you have 63 the Product fought.

DEMON. The Conftruction of the Table is plainly this: The 9 Digits being fet down in the upper Line, each of them is confidered as a Multiplicand; and is added to itfelf fucceffively, as oft as there are Units in every Digit not exceeding itfelf; and the Multipliers are fet in a Column on the fide againft the refpective Sums. For *Example,* the Multiplications of 6 from the upper Line are carried no farther than 6 on the fide; and fo of the reft. This explains the Reafon of the *Rule* when the greater Number is propofed as the Multiplicand; or where the Multiplicand and Multiplier are equal: But, where the leffer is propofed as the Multiplicand, yet we apply the greater to the Table, as if it was the Multiplicand, [which I fhall here fuppofe to be the fame in effect, and afterwards it fhall be demonftrated, *viz.* that 6 times 8 is the fame as 8 times 6, and fo of any other two Factors.]

CASE II. *To multiply any two Numbers into one another.*

Rule 1. Make any of the two Factors the *Multiplicand,* or *Multiplier;* but it will be generally moft convenient to make that the *Multiplier* which has feweft fignificant Figures: Then, tho' there's no matter in what order the Factors are fet down, yet, when it can be done, 'tis convenient to write the Multiplier under the Multiplicand; fo that the firft fignificant Figure on the Right-hand of this be over the firft fignificant Figure of the other, [whether thefe Figures be in the place of Units or not. See the Examples.] Then, in cafe there are o's ftanding on the Right-hand of either or both Factors, neglect them as if they were not there, and proceed thus:

2. Begin with the firft fignificant Figure of the Multiplier, and by it multiply every Figure of the Multiplicand, one after another, (by *Cafe* 1.) beginning at the firft fignificant Figure, and proceeding in order to the laft Figure on the left; and write down the Products. Thus, If the firft Product (or, that of the firft fignificant Figure of the Multiplicand) is lefs than 10, write it down; but if it exceeds 10, write down what's over any Number of 10's, and carry that Number (*i. e.* 1 for every 10) to the Product of the next Figure. If this Sum is lefs than 10, write it down; but if it exceeds 10, write down the Excefs of 10's, and carry the Number of 10's to the Product of the next Figure, and fo on: fetting the Figures to be written down all in a Line after one another, orderly, from the Right to the Left. Having thus gone thro' all the Figures of the Multiplicand, (not omitting the o's that ftand mixt with other Figures) write down the compleat Product of the laft Figure with the 10's of the preceding added to it.

F 2   3. Make

3. Make the same Operation upon the *Multiplicand*, with every significant Figure of the *Multiplier*; (passing all the o's) taking them in order as they stand towards the Left; and setting the several Products under one another: Thus, Set the first Figure of the Product made by every Figure of the *Multiplier*, at the same distance from the first Figure of the preceding Line, as their respective Figures in the *Multiplier* are; and place the rest of the Figures in order towards the Left, under the preceding.

4. Add all these particular Products into one Sum; taking them as they are set in the Columns under one another: And if the first significant Figure of both *Factors* was in the place of Units, this Sum is the Product sought (*Ex.* 1, 2, 3, 4, 5.) otherwise set as many o's before it as stand before the first significant Figure of both *Factors*, (as in *Ex.* 6, 7, 8, 9.)

I shall next apply this Rule in *Examples*.

Ex. 1.
642 Multiplicand.
4 Multiplier.
2568 Product.

To multiply 642 by 4; I say $4 \times 2 = 8$, which is set down; then $4 \times 4 = 16$, for which I write down 6 and carry 1; then 4×6 $(= 24) + 1 = 25$, which being written down, the Product is 2568.

Ex. 2.
85065 M^d.
8 M^r.
680520 Pr.

To multiply 85065 by 8, I work thus, $8 \times 5 = 40$, for which I write down o, and carry 4; then $8 \times 6 (= 48) + 4 = 52$, for which I write down 2 and carry 5; then $8 \times o (= o) + 5 = 5$, which I write down; then $8 \times 5 = 40$, for which I write down o, and carry 4; then, 8×8 $(= 64) + 4 = 68$; which being written down, the Product is 680520.

Ex. 3.
84653 M^d.
469 M^r.
761877 Prod. by 9.
507918 ———by 6.
338612 ———by 4
39702257 Total Prod.

To multiply 84653 by 469, I work thus; beginning with 9 (in the Units place of the Multiplier) I multiply by it the whole Multiplicand, in the manner of the preceding *Examples*. Then I take the next Figure of the Multiplier, 6, and by it in the same manner multiply the whole Multiplicand; setting the first Figure of this Product, 8, under the second Figure of the preceding, and the rest in order; making the same Operation with the next (and last) Figure of the Multiplier, *viz.* 4, and setting the first Figure of this Product (*viz.* 2) under the second of the preceding, and the rest in order. All these Products summed up, as you see in the *Ex.* gives the total Product.

Ex. 4.
6452
806
38712
51616
5200312

To multiply 6452 by 806: After multiplying by 6, I pass to 8; and because there is one o betwixt them, the first Figure of the last Line is set under the third place of the former, *i. e.* with one place betwixt the first Figures of the two Lines, because of the one o betwixt the Multipliers.

Ex. 5.
46007859
380046
276047154
184031436
368062872
138023577
1748510278514

To multiply 46007859 by 380046: After making the Product of 6 and 4, I pass to 8; setting its first Product two places distant from that of the preceding Line, because of the two o's betwixt 4 and 8.

In the four following Examples, wherein the firſt ſignificant Figure of each *Factor* is not in the place of *Units*, the Application of the Rule is ſo plain, that I need make no more words about it.

Examp. 6.	*Examp.* 7.	*Examp.* 8.	*Examp.* 9.
467	584000	46000	376890
2800	93	2700	5004000
3736	1752	322	150756
934	5256	92	188445
1307600	54312000	124200000	188595756000

In order to the Demonſtration of the preceding Rule, and for the ſake of ſome other ſpecial Rules following, we muſt premiſe theſe Truths, as belonging to the Theory of *Multiplication*.

L E M M A I.

If one Number is multiplied by another, the Product will be the ſame as if this other be multiplied by the former ; ī. e. Any one of the two Factors *may be made* -Multiplier *or* Multiplicand, *the Product will be the ſame.*

Examp. 4 times 7 = 7 times 4. A times B = B times A.

Demon. A ſmall Attention to the Idea of Numbers will make this Truth evident; and therefore few Writers think it needs any Demonſtration. However, as it is capable of one, and ſome may require it, I ſhall ſatisfy them. Thus;

Any Number B is only a certain Collection of *Units*; wherefore A times B is equal to A times each *Unit* in B, taken ſeparately and added together: but A times 1 is the ſame thing as A, or 1 time A, (from the Definition of Number:) Therefore A times each *Unit* in B is equal to A, (or 1 time A) taken as oft as there are *Units* in B ; *i. e.* B times A. Therefore A times B = B times A.

Or take this other more ſenſible Demonſtration.

Suppoſe any two Numbers, A, B ; let the *Units* of A be repreſented by a Row of Points, as in the Margin: Repeat this Row as oft as there are *Units* in B, and ſet them orderly under one another ; then it is plain, that there will be as many Columns of Points as there are Points or *Units* in A ; each of which Columns has as many Points as there are *Units* in B. Therefore the whole Number of Points which were at firſt made equal to B times A, (by repeating the Row A, B times) becomes neceſſarily equal to A times B.

A
. . . . .
B . . . . . &c.
. . . . .
&c.

L E M M A II.

If three or more Numbers are propoſed to be continually multipled, the laſt Product will ſtill be the ſame, in whatever Order the Factors *are taken.*

Examp. 3 × 5 × 7 = 3 × 7 × 5 = 7 × 3 × 5 = 7 × 5 × 3 = 5 × 3 × 7 = 5 × 7 × 3.

DEMON. C A S E I. If there are 3 *Factors* A, B, C ; then ſince the Product of 2 *Factors* is the ſame whichever of them is *Multiplier* ; *i. e.* A times B = B times A : Therefore the thing to be proved here is only, That whichever of the 3 *Factors* is caſt in the laſt place, the Product is ſtill the ſame, *viz.* that A B C = C B A = A C B. For ſince A B = B A ; therefore A B C = B A C. For the like Reaſon C B A = B C A, and A C B = C A B.

I

Now

Now then, to prove that $ABC = CBA = ACB$: First, taking the *Factors* in the Order A B C, expresses the Product of A by B, or B times A, and this Product A B taken C times; which makes C times B times A, or A taken C times B times; *i.e.* C B times A: So that $AB \times C = A \times CB$, or $CB \times A$; *i. e.* $ABC = CBA$. Again, A C B signifies A taken C times, and this Product A C taken B times; which is B times C times A; *i. e.* A taken B times C times; or, which is the same, C times B times: but this, *viz.* $A \times CB$ or $CB \times A$ was before shewn to be equal to $A B \times C$, therefore $ABC = CBA = ACB$.

CASE II. If there are more *Factors* than 3, then I prove the Truth proposed thus: I say, if it is true of any particular Number of *Factors* more than 2, it is therefore true if we take in one *Factor* more: but it is true of 3 *Factors*, as shewn above; therefore it is evidently true of 4, and hence again it is true of 5, and so on for ever. What is to be proved then, is the first Part, *viz.* the Connection of the Truth of the Rule for any Number of *Factors* more than 2, with the next Case, or one *Factor* more. Which I prove thus;

Let A B C D, &c. be a Product of any Number of *Factors*, in which it is supposed to be no matter in what Order they are taken; therefore I may cast any of its Terms last, the Product will still be equal, *viz.* $ABCD, \&c. = ABC, \&c. \times D = ABD, \&c. \times C = ACD, \&c. \times B = BCD, \&c. \times A$. Now, let another Term X be taken into the Question, then it is plain, that whatever *Factor* of the whole A, B, C, D, X, &c. we suppose to be last employed, the various Orders in which the preceding *Factors* may be employed, produce the same Effect, by supposition; wherefore all the various Orders of taking the whole *Factors*, wherein any particular *Factor* keeps the last place, produce the same Effect; because the various Orders of the preceding Terms have no different Effect by supposition. And therefore what remains to be proved is only this, *viz.* That the Products are all equal which are made by the several Orders wherein different *Factors* are put in the last place, which will easily appear thus: Since by supposition $ABCD, \&c. = ABC, \&c. \times D = ABD, \&c. \times C$, and so on, putting each *Factor* last: Therefore $ABCD, \&c. \times X = ABC, \&c. \times D \times X = ABD, \&c. \times C \times X$, and so on. But we may make X and the *Factor* preceding it, change places in each of these Expressions; the Product will still be the same by *Case* I. because it is a Product of 3 *Factors*, thus; $ABC, \&c. \times D \times X = ABC, \&c. \times X \times D$. Also, $ABD, \&c. \times C \times X = ABD, \&c. \times X \times C$, and so on; whereby each *Factor* is cast in the last place. Wherefore these Products are all equal, being each equal to the Product of another Order, and all these other Orders equal.

COROL. If two Numbers are proposed to be multiplied together, it is the same thing to do it by the *General Rule* at once; or, if one of the *Factors* is equal to the continual Product of two or more Numbers, then we may multiply the one given *Factor* first by one of these Numbers that produce the other, and then this Product by another of them, and so on thro' them all: Thus, $24 = 4 \times 6$. Therefore $62 \times 24 = 62 \times 4 \times 6$.

SCHOLIUM. What an *Aliquot Part* is, has been already explained. And from *Lem.* 1. it is evident, that if two Numbers are multiplied together, each of them is an *Aliquot Part* of the Product, and the other is the Denominator of the Part. So, because $4 \times 8 = 32$; therefore 8 is $\frac{1}{4}$ of 32, and 4 is $\frac{1}{8}$ of 32. Universally $A = \frac{1}{B}$ of AB, and $B = \frac{1}{A}$ of AB. Whence this follows, that if we multiply any Number by the *Aliquot Part* of another, and then this Product by the Denominator of that Part, the last Product is the same as if the first Number were multiplied by that other Number; which is a Truth manifest also from the nature of an *Aliquot Part*, without regard to this first *Lemma*. Again *observe*, That the Product of two Numbers is very naturally called, the *Multiple* of either of the *Factors*; and it is said to be a *Multiple* of it by the other *Factor*: So 48 is a *Multiple* of 8 by 6, or of 6 by 8. Also if two Numbers are multiplied by the same Number, the Products are called *Like-Multiples* of these Numbers, as 3×4 and 3×7 are *Like-Multiples* of 4 and 7. Now, as *Multiple* and *Aliquot Part* are directly opposite;

so

fo from the nature of them, this follows; That the Product of one Number by the *Aliquot Part*, or *Multiple* of another, is the like *Aliquot Part* or *Multiple* of the Product of thefe two Numbers. Thus; fuppofe $B = CD$; then is AC the $\frac{1}{D}$ Part of AB ($= ACD$,) and AB ($= ACD$) is the *Multiple* of AC by D.

L E M M A III.

If either or both of two Numbers are any how diftributed into two or more Parts; and if each Part of the one Number is multiplied into the other Number, or into each Part of the other; the Sum of all thefe Products is equal to the Product of the two given Numbers.

Examp. $3 + 4 = 7$, and $6 \times 7 = 3 \times 6 + 4 \times 6$. Again, $15 = 9 + 6$, alfo $8 = 3 + 5$. Therefore, $8 \times 15 = 3 \times 9 + 3 \times 6 + 5 \times 9 + 5 \times 6$. *Univerfally*, if $A = B + C$, then is $AD = BD + CD$; and if $D = E + F$, hence $AD = BE + BF + CE + CF$.

DEMON. The Reafon is manifeft; becaufe the Whole being nothing elfe than the Sum of all the Parts, when one Number or every Part of it is multiplied into every Part of another Number, then is the one Whole multiplied into the other; fo that the Sum of the Products made by the Parts muft be equal to the Product of the Whole.

D E M O N S T A T I O N of CASE II. *of the preceding* P R O B L E M.

I. When the firft fignificant Figure of each *Factor* is in the place of Units, (*Examp.* 1, 2, 3, 4, 5.) the Reafons of the Rule are thefe:

(1.) That either Number may be made *Multiplier* or *Multiplicand*, is demonftrated in *Lemma* 1. But making that one *Multiplier* which has feweft fignificant Figures, is moft convenient, becaufe it makes feweft partial Products.

(2.) If the *Multiplier* is a Digit, (*Ex.* 1, 2.) then by multiplying every Figure, *i. e.* every Part, of the *Multiplicand*, we multiply the Whole; and by writing down the Products that are lefs than 10, or the Excefs of 10's, in the places of the Figures multiplied, and carrying the Number of 10's to the Product of the next place, we do hereby gather together the fimilar Parts of the refpective Products; and fo do the fame thing in effect as if we wrote down the *Multiplicand* as oft as the *Multiplier* expreffes, and added them up: For the Sum of every Column is the Product of the Figure in the place of that Column, whereby it appears that the Addition, and the Multiplication according to this Rule, have the fame Effect: (fee *Ex.* 1.) Wherefore the Rule is right.

(3.) When there are more fignificant Figures than one in the *Multiplier*, as in *Ex.* 3, 4, 5. then, by applying them feparately, we refolve the *Multiplier* into Parts; and if the true Products made by each of thefe Parts multiplying the whole *Multiplicand*, are added together, the Sum is the Product fought, (by *Lem.* 3.) Again; the Product made by each Figure of the *Multiplier* taken in its fimple Value, is truly found by the Rule; as fhewn in the preceding Article; and by placing thefe Products with refpect to one another, fo as the firft Figure of each Product ftands under that Figure of the firft Product, (or Product of the Figure in the Units place of the *Multiplier*,) which is in the fame place as the multiplying Figure ftands in the whole *Multiplier*; thefe Products are added together according to the true Value they ought to have, by confidering the multiplying Figure in its complete Value. For *Examp.* The Product by the fecond Figure, or Place of 10's, is fet and added to the Product of the firft Figure, as if a Cypher or o had been prefix'd to it, whereby it is made 10 times as much as were the multiplying Figure in the place of Units, as it ought to be, fince that Figure is in the place of Tens, and not of Units; (fee *Schol.* after *Lem.* 2.) The fame Reafon holds in all the other Figures of the *Multiplier*. Therefore the Sum of the Products taken according to the Rule, is

the

the true Product fought. And the Reafon why we don't actually prefix thefe o's, is, becaufe it were fuperfluous as to the Sum. See thefe Examples.

Ex. 1. $6784 \times 3 = 20352$. For $6784 = 6000 + 700 + 80 + 4$. And the Operation by Parts is thus:

$$
\begin{array}{rl}
4 \times 3 = & 12 \\
80 \times 3 = & 240 \\
700 \times 3 = & 2100 \\
6000 \times 3 = & 18000 \\
\hline
& 20352
\end{array}
\qquad
\begin{array}{r}
6784 \\
\text{or thus,}\ 6784 \\
6784 \\
\hline
20352
\end{array}
$$

Example 2.

Common Way.	Or Thus.	
68749	68749	
853	853	
206247	206247	Prod. by 3.
343745	3437450	Prod. by 50.
549992	54999200	Prod. by 800.
58642897	58642897	Total Prod.

II. When the firft fignificant Figure of either or both *Factors* does not ftand in the place of *Units:* Then,

(1) When it is fo only in one of the *Factors,* (as *Ex.* 6, 7.) then by taking that *Factor* as if the firft fignificant Figure were in the place of *Units;* i. e. neglecting all the o's that ftand on the right, we do indeed work only with the 10th or 100th, &c. part of it; therefore the total Product found is accordingly but the 10th or 100th Part of what it ought to be: and therefore to have the true Product, we ought to multiply the laft Product by 10, 100, &c. (by *Schol. Lem.* 2.) which is done by fetting as many o's before the Product as were before the Multiplier.

(2.) When it is fo in both *Factors* (as in *Ex.* 8, 9.) there is the fame Reafon for fetting before the Product the o's that belong to the one *Factor,* as thofe belonging to the other: For after correcting the Product by the o's of the one *Factor,* it wants to be corrected again by thofe of the other: Therefore when there are o's belonging to both, they ought all to be fet before the Product.

Therefore this *Rule* is true in all poffible Cafes.

PROOF of MULTIPLICATION.

In the firft place, we muft obferve, That the multiplying of one Digit by another has no other Proof than by Addition: But the Table being examin'd and found true, we are to depend upon that or Memory for thefe fimple Operations; the Proof here defigned being for other Cafes where either or both *Factors* exceed 9: And this may be done feveral ways. Thus,

1. By *Multiplication* itfelf: For if we change the *Factors,* and make that the *Multiplier* which was before the *Multiplicand,* we ought to have the fame Product, by *Lem.* 1.

2. By *Divifion:* But this cannot be applied till that *Rule* is learned.

But both thefe Methods are too tedious to be of Ufe; and therefore,

3. The moft convenient and eafy Proof is by help of the Number 9, like what we have in *Addition;* which is performed thus:

Caft

Caſt out the 9's out of the two *Factors* (in the manner taught in the like *Proof of Addition*) and mark the Number that is under or over 9's in each; if either of them is 0, paſs immediately to the Product, and caſt out the 9's of it; the Exceſs ought here to be 0, which makes the Proof. But if the Number under or over 9's is not 0 in either Factor, then multiply theſe two Numbers together, and mark alſo what is under or over 9's in their Product; this Number and what is over 9's in the total Product of the Example ought to be equal.

Ex. 1.		

<div>

Ex. 1.

```
  476
   68
 ----
 3808
 2856
 ----
 3 ∴ 8
```

In this firſt Example, the Exceſs of 9's in the *Multiplicand* is 8, in the *Multiplier* it is 5; then 8 × 5 = 40, and ſo the Exceſs of 9 is 4; exactly equal to the Exceſs of 9's in the Product 32368.

Ex. 2.

```
   87
   36
 ----
  522
  261
 ----
 3132
```

In the ſecond Example, the Exceſs of 9's in the *Multiplier* is 0, therefore I paſs immediately to the Product, where I find it is alſo 0.

</div>

Demonſtration. The Reaſon of the Practice of caſting out the 9's in the ſeveral Numbers has been already demonſtrated in *Addition*; and the Reaſon of the reſt of the Work will eaſily appear, thus: 1. If either *Factor* is a preciſe Number of 9's, (*i. e.* when there is no Exceſs) as in Example 2. it is plain the Product muſt be ſo too, for it is only that Number or 9's taken a number of times. But, 2. If each of them is equal to a Number of 9's, and ſome leſſer Number over, then let us repreſent them thus; Let one be A + *b*, where A repreſents any Number of 9's, and *b* the Number over: Let the other be N + *c*, where N is any Number of 9's, and *c* the Number over. But now, by *Lem.* 4. the Product of theſe *Factors* is equal to the Sum of the Products of all the Parts of the one by all the Parts of the other; and ſo the Product is here $\overline{A \times N}$, + $\overline{A \times c}$, + $\overline{N \times b}$, + $\overline{b \times c}$. But the firſt three Products are each a Number of 9's, becauſe one of their *Factors* is ſo; therefore theſe being caſt away, there remains only $\overline{b \times c}$. And if the 9's are alſo caſt out of this, the Exceſs is the Exceſs of 9's in the total Product; but *b*, *c* are the Exceſs in the *Factors*, and $\overline{b \times c}$ their Product: therefore the Rule is true.

SCHOLIUM. The Objection made to this *Proof* is the ſame as that mentioned already againſt the like *Proof for Addition*; and therefore the ſame Anſwer ſerves here.

§. 2. Containing

PARTICULAR METHODS *of working* MULTIPLICATION *in certain Caſes; whereby it is either contracted into a ſhorter Work than the general Rule, or made eaſier and more certain with as large a Work; and, in ſome Caſes, with a little more Work.*

THE firſt five CASES are ſuch wherein the very ſame Operation is performed as by the *General Rule*; only the Trouble and Time of writing down a great many Figures is ſaved.

CASE I. When one of the *Factors* has any Figure whatever in the place of *Units*, and 1 in all the reſt, as 16, 114, 1118; make that the *Multiplier*, and the Product may be got all at once thus;

(1.) Suppoſe the number of Places of the *Multiplier* do not exceed thoſe of the *Multiplicand*, then multiply by the Figure in the Units place of the *Multiplier*, and with every Product, till you come to the Product of that Figure that ſtands over the laſt 1 of the *Multiplier*, (*i. e.* till you have multiplied as many Figures as the Number of Places in the *Multiplier*) add in all the preceding Figures on the right of the *Multiplicand*; to the next Product after this, add in all the preceding except the firſt, (or Units place) and at every

ſuc-

fucceeding Product exclude always one more in order from the right, [and when you come to thefe exclusions, it will be fit to fet a Point over the Figure which is to be excluded at the next Operation,] *viz.* the remoteft on the right of thefe that were taken in at the laft ftep. Write down at every ftep what's under or over 10's, and carry the 10's to the next ftep; and when you have gone thro' all the Figures of the *Multiplicand,* to the number of 10's carried from the laft ftep add in all the Figures of the *Multiplicand* after the laft Point, pointing alfo the laft you take in at this ftep; then take in all from the laft Point, and fo on till you take in the very laft Figure alone; and thus you have the true Product fought; as in Examples 1, 2, 3.

(2.) If the Places of the *Multiplier* exceed in Number thofe of the *Multiplicand,* do all as in the former *Cafe*; only, when all the Figures of the *Multiplicand* are multiplied, the whole of them muft be fumm'd and taken in as oft, and once more, as the difference betwixt the Number of Places of the *Multiplicand,* and the Number of 1's in the *Multiplier,* (*i. e.* plainly till you have got a product Figure under every 1 of the *Multiplier :*) then beginning ftill at the laft place, take in one place fewer, (as before) till the laft place is taken in alone. See *Example* 4.

The Reafon of this Practice will eafily appear by comparing a few Examples wrought this way, and alfo the common way.

Example 1.

```
 4658      4658
   16        16
 ────      ────
74528     27948
           4658
          ────
          74528
```

Example 2.

```
 5276      5276
  114       114
 ────      ────
60164     21104
           5276
           5276
          ────
          60164
```

Example 3.

```
 263487     263487
   1113       1113
 ──────     ──────
293261031    790461
            263487
            263487
            263487
           ────────
           293261031
```

Example 4.

```
  847        847
11113      11113
─────      ─────
9412711     2541
            847
            847
            847
           847
          ───────
          9412711
```

The Operations of thefe Examples are thus; *viz.*

Examp. 1. 6 × 8 = 48, which is 8 and carry 4; then 6 × 5 (= 30) + 4 carried (= 34) + 8 (the next Figure on the right) = 42; which is 2, and carry 4. Then 6 × 6 (= 36) + 4 carried (= 40) + 5 (on the right) = 45; which is 5, and carry 4. Then 6 × 4 (= 24) + 4 carried (= 28) + 6 (on the right) = 34; which is 4, and carry 3; then 3 (carried) + 4 (the laft Figure) = 7.

Examp. 2. 4 × 6 = 24; which is 4, and carry 2 : then 4 × 7 (= 28) + 2 carried (= 30) + 6 (on the right) = 36; which is 6, and carry 3 : then 4 × 2 (= 8) + 3 carried (= 11) + 7 + 6 (the two next on the right) = 24; which is 4, and carry 2 : then 4 × 5 (= 20) + 2 carried (= 22) + 2 + 7 (on the right) = 31; which is 1 and carry 3 : then 3 carried

ried $+5+2$ (on the right) $=10$; which is 0, and carry 1 : then 1 (carried) $+5$ (on the right) $=6$.

Examp. 3. Here I shall spare repeating what is set down and carried; or the words *carried*, and *on the right*, becaufe you know how to fupply them. The Work is thus: $3\times7=21$; then $3\times8(=24)+2+7=33$; then $3\times4(=12)+3+8+7=30$; then $3\times3(=9)+3+4+8+7=31$; then $3\times6(=18)+3+3+4+8=36$; then $3\times2(=6)+3+6+3+4=22$; then $2+2+6+3=13$; then $1+2+6=9$; then (0 carried $+$) 2.

Examp. 4. Thus $3\times7=21$; then $3\times4(=12)+2+7=21$; then $3\times8(=24)+2+4+7=37$; then $3+8+4+7=22$; then $2+8+4+7=21$; then $2+8+4=14$; then $1+8=9$.

SCHOLIUM. To compare thefe Operations with the fame Examples at large, *obferve* what Figures of the *Multiplicand* (as it ftands written down for every 1 in the *Multiplier*) ftand under each Figure of the firft product Line; and thefe fhew the reafon of taking in the Figures on the Right-hand of the *Multiplicand*, and how this Rule was invented.

I advife a Learner to make himfelf familiar with the Practice of Examples like the firft; becaufe they occur frequently in common Bufinefs.

Alfo if the *Multiplier* has but two Places, tho' 2 is in the fecond place, the Product may eafily be made all at once, by taking in with every Product made by the place of Units, double the Figure on the Right-hand. Practice will make this eafy, and it will be very ufeful. The like Method may be ufed whatever Figure is in the fecond Place, (by taking in with every Product as many times the preceding Figure:) but the greater that Figure is, it is the more difficult; and I would only recommend the Practice for 2; unlefs the *Multiplicand* have not above 3 or 4 Places, and thefe alfo fmall Figures; for then we may ufe this Method with 3, 4, or 5 in the fecond Place.

$\begin{array}{r} 546 \\ 28 \\ \hline 15288 \end{array}$
The *Example* annex'd is thus done; $8\times6=48$; then $8\times4(=32)+4+12$ $=48$; then $8\times5(=40)+4+8=52$; then $5+10=15$.

Note, If 1 is in all the Places, the Practice is fo much the eafier, without altering the Rule.

CASE II. If one of the *Factors* has a fignificant Figure in the Place of Units, 1 in the higheft Place, and 0's betwixt them; make that in Units place the *Multiplier*, and work thus.

1. Suppofe the places of the *Multiplier* do not exceed in number thofe of the *Multiplicand*; then multiply by the firft Figure; and when you are come to the Figure of the *Multiplicand* that ftands over the 1 in the *Multiplier*, (*i. e.* having made as many Products as there are Places in the *Multiplier*) with that Product take in the firft Figure of the *Multiplicand*; and with every fucceeding Product add in the next Figure in order, (and it will be convenient to put a Point over every Figure when taken in, that you may more readily know what is to be next taken in.) After the laft Figure of the *Multiplicand* is multiplied, fet down all the remaining Figures of the *Multiplicand* that are on the left of the Figure laft taken in, after having added to them what was carried from the laft Product.

The Reafon of this is obvious in the following Examples.

Example 1.			Example 2.	
	57648		4653	4653
57648	103		1004	1004
103	172944		4671612	18612
5937744	57648			4653
	5937744			4671612

2. Sup-

2. Suppose the Places in the *Multiplier* exceed those in the *Multiplicand*; then if they exceed only by one, there is no place for a Contraction, (*Examp.* 3, 4.) But if there is only one Place more, it may receive a small Contraction thus; Add what's carried at the laft Product to the whole *Multiplicand*, and fet down the Sum on the left of the Figures already fet down; (*Examp.* 5.)

Example 3.	Example 4.	Example 5.	
3207	468	8674	8674
100005	1000007	10006	10006
320716035	468003276	86792044	52044
			8674
			86792044

Observe, If there are any more than one 1 upon the Left-hand of the o's, this will be but a Mixture of this and the preceding Cafe, to be done thus; *viz.* After you begin to take in the Figures of the *Multiplicand,* take them in gradually, firft one, then two, and fo on, till you take in as many as the number of 1's in the *Multiplier,* (beginning ftill at the firft Figure of the *Multiplicand*) and after that (taking in ftill the fame Number of Figures, begin one place nearer the Left-hand; and when you have not as many Figures to take in, take in all you have. The following Examples, without any more Words, will fufficiently explain this.

Example 6.		Example 7.
853467	853467	67853945
11004	11004	111007
9291550868	3413868	7532262872615
	853467	
	852467	
	9291550868	

Example 8.	Example 9.	Example 10.	Example 11.
4632	463	463	463
111004	11004	110004	1100004
514170528	5094852	50931852	50 331852

CASE III. When one of the *Factors* has any Figure greater than 1 in the higheft Place, and 1 in all the other Places make that Figure the *Multiplier* : then work thus, *viz.* Take the Figure in Units place of the *Multiplicand,* and fet it down; next take the Sum of the firft two places, and then the Sum of the firft three Places, and fo on (beginning ftill at the firft Place) till you have made as many fuch Operations as the Number of times that 1 is in the *Multiplier,* (and if there are not as many Figures in the *Multiplicand* as to have a Figure more to take in at every Operation, you muft continue to take in the whole *Multiplicand* till your number of Operations are compleat) ftill writing down the Sums as in *Addition,* and carrying the 10's. After this, take the Figure in the higheft Place, and by it multiply the whole *Multiplicand* in order; taking into the firft Product what was carried from the preceding Operations; and to this and each other Product add the Sum

of

of as many of the next following Figures on the left as the Number of 1's in the *Multiplier*; and if there are not as many, add all that are.

Note, It will be convenient to set the higheft Place of the *Multiplier* under the firft of the *Multiplicand.*

Example 1.		*Example 2.*		*Example 3.*	
4768	4768	7236	7236	86	86
41	41	4111	4111	4111	4111
195488	4768	29747196	7236	353546	86
	19072		7236		86
	195488		7236		86
			28944		344
			29747196		353546

CASE IV. When one of the *Factors* has 1 in the firft Place, any Figure in the higheft Place, and o's betwixt them, make that Figure the *Multiplier*, and work thus; *viz.* Write down as many of the firft places of the *Multiplicand* as are in Number one lefs than the Places in the *Multiplier*, (*Examp.* 1, 2, 3.) and if there are not as many, make it up with o's fet on the left, (*Examp.* 4.) After this, multiply by the laft Figure of the *Multiplier*, taking with every Product the Figure of the following Place but one or two, &c. according to the Number of o's in the *Multiplier*, (and fet a Point over the Figures taken in, which will be a guide to the next) and you have the Product fought.

Examp. 1.		*Examp. 2*		*Examp. 3.*	*Examp. 4.*
678	678	34675	34675	496	7263
301	301	3001	3001	8001	400001
204078	678	104059675	34675	3968496	2905207263
	2034		104025		
	204078		104059675		

CASE V. When one of the *Factors* confifts of the fame Figure in all its Places (as 66, or 444,) make that the *Multiplier*, and work thus; *viz.* Multiply by that Figure, and out of this Product make up the total Product in this manner: Begin at the Place of Units, and firft take one Figure, then two, then three, &c. (fetting down the Sums under or over 10's, and carry the 10's;) repeating the Operation ftill from the firft Place, as oft as the Number of Places in the *Multiplier*; and if there is not another Figure to take in at every Operation, you muft continue to take in the whole *Multiplicand*, fo oft till your Number of Operations be finifhed. Then begin at the fecond place, and third place, &c. fucceffively; taking in from each as many on the left as the Number of Places of the *Multiplier*, as long as you can find as many; and when they fail, take in all that remains till the laft Figure is taken in alone; (minding always to carry the 10's from every Operation to the next.)

Examp.

Example 1.

$$
\begin{array}{r}
8739 \\
444 \\
\hline
34956 \text{ Prod. by } 4 \\
3880116 \text{ Total.}
\end{array}
$$

$$
\begin{array}{r}
8739 \\
444 \\
\hline
34956 \\
34956 \\
34956 \\
3880116
\end{array}
$$

Example 2.

$$
\begin{array}{r}
46 \\
33333 \\
\hline
138 \\
1533318
\end{array}
$$

$$
\begin{array}{r}
46 \\
33333 \\
\hline
138 \\
138 \\
138 \\
138 \\
138 \\
1533318
\end{array}
$$

SCHOLIUM. If the *Multiplier* confifts of 1 in all its places, the Practice is the fame; only we have not any previous Multiplication, the fecond part of the Work being made upon the *Multiplicand* itfelf.

COROL. When the *Multiplier* confifts of any other Figures than 1, we may do the Work thus: Firft find the Product, as if the *Multiplier* confifted of as many 1's, and then multiply this Product by the Figure of the given *Multiplier*; fo the preceding *Example* will ftand thus:

$$
\frac{8739 \times 444}{970029} \text{ Prod by } 111.
$$
$$
\frac{4}{3880116}
$$

The Reafon of this you'll find at *Cafe* 9. joined with this, that $444 = 111 \times 4$.

CASE VI. If the *Multiplier* is a Number which has 9 in all its places, as $9, 99, 999, &c.$ fet as many o's on the Right-hand of the *Multiplicand*, and fubtract it from itfelf fo increafed: the Remainder is the Product fought, as in thefe Examples.

Example 1.
$$
468 \times 9 = 4212
$$
Operation.
$$
\begin{array}{r}
4680 \\
468 \\
\hline
4212
\end{array}
$$

Example 2.
$$
3726 \times 99 = 36884
$$
$$
\begin{array}{r}
372600 \\
3726 \\
\hline
368874
\end{array}
$$

Example 3.
$$
7568 \times 999 = 7560432
$$
$$
\begin{array}{r}
7568000 \\
7568 \\
\hline
7560432
\end{array}
$$

The *Reafon* of this Rule is obvious, for by the o's prefix'd to the *Multiplicand*, it is multiplied by a Number which exceeds the given *Multiplier* by 1, (fo $10 - 1 = 9$, $100 - 1 = 99$, and fo on.) Wherefore the *Multiplier* being fubtracted from this Product, the Remainder muft be the true Product fought.

Obferve, This Subtraction may eafily be perform'd without the trouble of writing the *Multiplicand* ofiner than once, by imagining the o's that ought to be prefix'd: Thus, take the firft Figure of the *Multiplicand* from 10, then the next Figure increafed by 1, from 10, and fo on, till you have made as many Subtractions from 10, as the Number of 9's in the *Multiplier*: then you fubtract from the Figures of the *Multiplicand* itfelf; and when there are no more Figures to fubtract, write down all the remaining Figures from which no Subtraction has been made, after fubtracting 1 from them when 10 was borrowed in the preceding Subtraction. The preceeding *Examples* fufficiently fhew how this is to be done.

Obferve again, To multiply any Number confifting all of 9's by itfelf; the Product has 1 in the place of Units, then after it a Number of o's fewer by one than there are 9's in the Number multiplied, then 8, and laftly, after it as many 9's as there are o's before it.

Ex. $999 \times 999 = 998001$, and $99999 \times 99999 = 9999800001$.

The

The *Reason* of this will appear to be universal, by confidering one *Example* done in the manner of the preceding Rule; thus,

$$
\begin{array}{r}
9999900000 \\
99999 \\
\hline
9999800001
\end{array}
$$

CASE VII. To multiply any Number by 5; multiply it firft by 10, *i. e.* fet (or fuppofe) o before it; then take the half of it.

Ex. 1. 6458×5
$\qquad$
32290 Prod.

Ex. 2. 7287×5
$\qquad$
36435 Prod.

SCHOLIUM. I have here propofed a Divifion, tho' that Rule is not yet taught: But it was fit to put all together that relates to Multiplication, tho' the Learner fhould refer this Cafe till he has Divifion; yet I think it will be eafy to fee how any Number is halved, by confidering thefe two Examples. And as to the Ufe of this Rule, it's certainly eafier than multiplying by 5, tho' that is eafy itfelf.

GENERAL COROLLARY *to the preceding 7 Cafes.* If the Parts of the *Multiplier* coincide with any of the preceding Cafes, we may apply them feparately: As in thefe Examples.

Ex. 1.
$$
\begin{array}{r}
7468 \\
159 \\
\hline
67212 \\
112020 \\
\hline
1187412
\end{array}
$$

Ex. 2.
$$
\begin{array}{r}
367845 \\
4118 \\
\hline
6621210 \\
15081645 \\
\hline
1514785710
\end{array}
$$

For *Ex.* 1. Work by 9 (as in Cafe 6) then for 15 (as Cafe 1.)

For *Ex.* 2. Work with 18, and 41 (by Cafe 1. and 3.)

CASE VIII. If one of the *Factors* confifts of two Places, and is equal to the Product of any two Digits, as $28 = 4 \times 7$; multiply by one of thefe Digits, and the Product found by the other; and the laft Product is that fought.

Ex. 7264×28
$$
\begin{array}{r}
4 \\
\hline
29056 \\
7 \\
\hline
203392 \text{ Prod.}
\end{array}
$$

The Reafon of this is obvious, for 7 times 4 times is 28 times. Or may be deduced from *Lemma* 3. *Corol.* For having multiplied by 4, which is only the 7th part of 28, I muft multiply again by 7 the Denominator of the Part, to make the true Product by the whole 28.

SCHOLIUM. If the *Multiplier* is equal to the *Product* of any three or more Digits; we may take the fame Method, by multiplying continually by all thefe Digits: But this will not in every Cafe afford any Compend; nor will it always appear eafily what Digits will produce the given Number: However the Reafon of the Practice is evident from *Lem.* 3. *Cor.* For $7486 \times 8 \times 9 \times 6 = 8 \times 9 \times 6 \times 7486. = 432 \times 7486$, becaufe $8 \times 9 \times 6 = 432$.

Ex. $7486 \times 432 \; (= 8 \times 9 \times 6)$
$$
\begin{array}{r}
8 \\
\hline
61488 \\
9 \\
\hline
553392 \\
6 \\
\hline
3320352 \text{ Product.}
\end{array}
$$

CASE IX. In Multiplication of great Numbers, one part of the hazard of erring proceeds from the too great Diftance betwixt the Figures of the feveral Product Lines and the correfponding Figures of the *Multiplicand*; and alfo their not ftanding directly under one another, fo that we muft always look athwart from the one to the other.

In many Cafes, tho' we cannot contract the Work, either as to the Operation or Number of Figures written, yet we may make it more fimple and eafy, by help of Addition or Subtraction, or more fimple Multiplication; and fometimes with thefe Methods we may

mix

mix fome of the former. The following Examples will fufficiently inftruct you how to do the like in other Cafes: For the different Circumftances of the *Multiplier* makes the Variety unlimited, and therefore there can be no general Rule.

Ex. 1.	648	Ex. 2.	467		467	Ex. 3.	57684	Ex. 4.	974
	76		78		78		63		862
	3888		3736	Or thus	3269		173052		1948
	4536		3269		3736		346104		5844
	49248		36426		36426		3634092		7792
									839588

Ex. 5.	5689	Ex. 6.	4956		4956	Ex. 7.	5648	Ex.8.	724689
	286		168		168		459		545
	11378		39648	Or thus	79296		50832		3623445
	45512		79296		39648		254160		3661005
	34134		832608		832608		2592432		398955505
	1627054								

The Operation of thefe Examples is thus:

Ex. 1. After multiplying by 6, I add that Product to the *Multiplicand,* inftead of multiplying by 7.

Ex. 2. After multiplying by 8, I fubtract the *Multiplicand* out of this Product, inftead of multiplying by 7. Or, according to the other Method, I firft multiply by 7, and to this Product add the *Multiplicand.* The reafon of placing the two Lines as you fee them ftand here, is obvious.

Ex. 3. After multiplying by 3, I double this Product for 6.

Ex. 4. I multiply by 2, then multiply this Product by 3 (for 6) then add thefe two Lines (the firft Figure of the one to the firft of the other, and fo on) for 8.

Ex. 5. I multiply by 2, this Product by 4, (for the 8) and then fubtract the firft Line from the fecond (for the 6) or multiply the firft Line by 3.

Ex. 6. I multiply by 8, and double this Product (for the 16.) Or multiply by 16 and halve this Product (for the 8.)

Ex. 7. I multiply by 9, and becaufe $9 \times 5 = 45$, I multiply the laft Product by 5, either the common way, or by *Cafe* 7.

Ex. 8. I multiply by 5 (*Cafe* 7.) for the laft 5; and this Product by 9 (*Cafe* 6.) for the 45.

An UNIVERSAL METHOD *for all Cafes; whereby, tho' there is no Contraction, and even fome more to do, yet it makes the Work fo eafy, that there is no time loft, at leaft in large Examples, and more Certainty in the Operation.* Thus:

Write down the *Multiplicand,* then double it; add this Sum to the *Multiplicand,* and this again, and fo on, every Sum to the *Multiplicand,* till you have nine Numbers. And it's plain that thus you have a Table of the Products of the *Multiplicand* by all the Digits; made up by a very fimple and eafy Operation: And then you have no more to do, but transfer your feveral Products out of this Table, and fum them up.

TABLE.

TABLE.

1	467853798
2	935707596
3	1403561394
4	1871415192
5	2339268990
6	2807122788
7	3274976586
8	3742830384
9	4210684182

Example.

$$
\begin{array}{r}
467853798 \\
6839754 \\
\hline
1871415192 \\
2339268990 \\
3274976586 \\
4210684182 \\
1403561394 \\
3742830384 \\
2807122788 \\
\hline
3200004886285692
\end{array}
$$

SCHOLIUMS.

1. This Method is univerfal; but we need not apply it in every Cafe, for that would not always be beft: But in fuch Examples as this, I think the Eafe and Readinefs with which it's done, does more than fave the time fpent in making the Table; with this Advantage, that the Work is perform'd with much more Certainty, becaufe it's more fimple.

2. Again this may be contracted in many Cafes: for there is no neceflity always to make the Table for all the nine Digits. And it may happen, that by help of fome of the preceding Methods, we can as eafily make a Table for few, or no more than we have ufe for in the *Multiplier;* nor is it any great matter in what order they ftand in the Table.

Ex. 1. 78659 by 6897. In making this Table, I firft take 3; then double its Product for 6; and do the reft by the common way.

Table for *Ex.* 1.

1	78659
3	227977
6	455954
7	534613
8	613272
9	691932

Ex. 2. 783596 by 3856. Firft I multiply by 3, then by 5, (as in *Cafe* 7.) then add thefe Products for 8, then fubtract the firft from this for 7.

Table for *Ex.* 2.

1	783596
3	2350788
5	3917980
8	6268768
7	5485172

MULTIPLICATION by *NEPER's* RODS.

The great and everlafting Honour of our Country, the Lord *Neper*, confidering the vaft Advantage of the preceding Method of *Tabulating* the *Multiplicand*, to make it yet more eafy, contrived the following Machine, for the more certain and ready way of making the Table: To underftand which, we muft fet before us what they call,

PYTHAGORAS's Table of *Multiplication.*

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The Conftruction of this Table is the fame as that fhewn in *Problem* 1. with this difference, that here there is a compleat Column of Products from every Digit on the head, to 9 times that Digit; fo that either of the Factors may be found on the head, or on the left fide.

Now, fuppofe this Table to be made upon a Plate of Metal, Ivory, Wood, or Paft-board; and then conceive the feveral Columns (ftanding downwards from the Digits on the head) to be cut afunder, and thefe are what we call *Neper's* Rods, for *Multiplication.* But then there muft be a good number of each; for as many

H   times

times as any Figure is in the *Multiplicand*, so many Rods of that Species, (*i. e.* with that Figure on the Top of it) muft we have; tho' 6 Rods of each Species will be fufficient for any Example in common Affairs. There muft alfo be as many Rods of o's. But before we explain the way of ufing thefe Rods, there is another thing to be known, *viz.* That the Figures on every Rod are written in an Order different from that in the Table; *Thus,* The little fquare Space or Divifion in which the feveral Products of every Column are written, is divided into two Parts by a Line a-crofs from the upper Angle on the Right to the lower on the Left; and if the Product is a Digit, it is fet in the lower Divifion; if it has two Places, the firft is fet in the lower, and the fecond in the upper Divifion; but the Space on the Top is not divided. Alfo there is a Rod of Digits not divided, which is called the *Index-Rod*; and of this we need but one fingle Rod. See here the Figure of all the different Rods, and the Index, feparate from one another.

NEPER's RODS.

Index Rod.	1	2	3	4	5	6	7	8	9	0
1	1	2	3	4	5	6	7	8	9	0
2	2	4	6	8	10	12	14	16	18	0
3	3	6	9	12	15	18	21	24	27	0
4	4	8	12	16	20	24	28	32	36	0
5	5	10	15	20	25	30	35	40	45	0
6	6	12	18	24	30	36	42	48	54	0
7	7	14	21	28	35	42	49	56	63	0
8	8	16	24	32	40	48	56	64	72	0
9	9	18	27	36	45	54	63	72	81	0

USE of the RODS.

Lay down firft the Index-Rod; on the Right of it, fet a Rod on whofe Top is the Figure in the higheft Place of the *Multiplicand*; next to this again fet the Rod on whofe Top is the next Figure of the *Multiplicand*; and fo on in order to the firft Figure: Then is your *Multiplicand* tabulated for all the 9 Digits; for in the fame Line of Squares ftanding againft every Figure of the Index-Rod, you have the Product of that Figure; and therefore you have no more to do but transfer the Products, and fum them.

But in taking out thefe Products from the Rods, the Order in which the Figures ftand obliges you to a very eafy and fmall Addition, thus; Begin to take out the Figure in the lower Part (or *Units* place) of the Square of the firft Rod on the Right; add the Figure in the upper Part of this Rod to that in the lower Part of the next, and fo on; which may be done as faft as you can look upon them. To make this Practice as clear as poffible, take this Example.

Examp.

The Rods set together for the Number 4768.

1	4	7	6	8
2	0/8	1/4	1/2	1/6
3	1/2	2/1	1/8	2/4
4	1/6	2/8	2/4	3/2
5	2/0	3/5	3/0	4/0
6	2/4	4/2	3/6	4/8
7	2/8	4/9	4/2	5/6
8	3/2	5/6	4/8	6/4
9	3/6	6/3	5/4	7/2

Examp. Multiply 4768 by 385.

Againſt 5 in the Index I find this Number, according to the Rule,	23840
Againſt 8 this Number, — —	38144
Againſt 3 this Number, — —	14304
Total Product,	1835680

To make the Uſe of the *Rods* yet more regular and eaſy, they are kept in a flat ſquare Box, whoſe Breadth is that of ten Rods, and the Length that of one Rod; as Thick as to hold ſix, (or as many Rods as you pleaſe;) the Capacity of the Box being divided into ten Cells, for the different Species of Rods. When the Rods are put up in the Box, (each Species in its own Cell, diſtinguiſhed by the firſt Figure of the Rod ſet before it on the Face of the Box near the Top,) as much of every Rod ſtands without the Box as ſhews the firſt Figure of that Rod: Alſo upon one of the flat Sides, without and near the Edge upon the Left-hand, the Index-Rod is fixed; and along the Foot there is a ſmall Ledget; ſo that the Rods when applied are laid upon this Side, and ſupported by the Ledget, which makes the Practice very eaſy. But in caſe the *Multiplicand* ſhould have more than 9 Places, that upper Face of the Box may be made broader.

Some make the *Rods* with four different Faces, and Figures on each, for different purpoſes. But I have explained what is neceſſary for *Common Multiplication*, and ſhall leave it.

A NEW METHOD by a ſmall Moveable TABLE.

To Thoſe who want *Rods*, I propoſe the following Methods, which, with the help of *Caſe* VI. and VII. and ſome Hints given in *Caſe* X. may, I believe, be near as eaſy and expeditious as the *Rods.* Thus:

1. Make a Table of the *Multiplicand* only for the Numbers 1, 2, 5; (uſing *Caſe* VII. for 5.) and make it upon a bit of looſe Paper, that it may be always applied directly and immediately over the Place where every particular Product is to be written down, (for much of the difficulty lies, as I have already obſerved, in the Diſtance and croſs Poſition of the *Multiplicand* to the ſeveral Products,) and out of this ſmall Table find your Product thus:

Suppoſe for a Multiplicand 685497.

TABLE.

1	685497
2	1370994
5	3427485

When the Figure of the Multiplier is 2 or 5, here you have the Products; then for 3 add 2 to 1, (i. e. the Numbers againſt 2 and 1;) for 4, double the Number againſt 2. For 6, add 5 and 1; or multiply 2 by 3. For 7 add 5 and 2. For 8 add 5, 2 and 1; for 9, uſe the Method of *Caſe* VI.

2. We

2. We may alſo make the Table for 3 and 5, as here: And then for 2, double 1; for 4, add 3 and 1; for 6, double 3, or add 5 and 1; for 7, double 3 and add 1; that is, to every Product add the Figure of the *Multiplicand.* For 8, add 5 and 3; for 9, uſe *Caſe* VI.

1	685497
3	2056491
5	3427485

3. Or alſo make the Table for 3 and 7: And to get the 7, I multiply 3 by 2, and add 1. Then in uſing the Table, to get the Product by 4, add 2 to 1. For 5, uſe *Caſe* VII. For 6, double 2; for 8, add 1 and 7; for 9, uſe *Caſe* VI. or multiply 3 by 3.

1	685497
3	2056491
7	4798479

4. Or we may make the Table for 1, 2, 4. Then for 3, add 1 and 2; for 5, uſe *Caſe* VII. or add 1 and 4. For 6, add 2 and 4; for 7, add 1, 2, and 4; for 8, double 4; for 9, uſe *Caſe* VI.

1	685497
2	1370994
4	2741988

5. Or laſtly, make the Table for 1, 3, 4. Then for 2, double 1; for 5, uſe *Caſe* VII. for 6, double 3; for 7, add 4 and 3; for 8, double 4; for 9, uſe *Caſe* VI.

1	685497
3	2056491
4	2741988

Uſe any of theſe Tables you pleaſe; and for different Multipliers, one of them may, perhaps, be preferable to another. But if all the 9 Digits are in any Multiplier, it is indifferent which of them you chuſe; tho' I think the third Method has the Advantage.

CHAP. VI.

DIVISION of *Abſtract Whole* NUMBERS.

DEFINITION.

DIVISION *findeth how oft one* Number *is contained in another.*
The Number divided, (or which is conſider'd as the containing Number) is called the *Dividend*; the Number dividing, (or which is conſidered as contained) is called the *Diviſor*; and the Number ſought, is called the *Quotient*, or *Quote*, (from *Quoties, How oft:*) becauſe it ſhews the *how oft* ſought. *Examp.* If we enquire how oft 3 is contained in 12, the Anſwer is 4 times: And 12 is the *Dividend*; 3 the *Diviſor*; 4 the *Quote.*

SCHOLIUMS.

1. Every greater Number is not a *Multiple* of every leſſer: therefore when a greater Number is propoſed to be divided by a leſſer, and is not a Multiple of it, this Operation finds how oft the Diviſor is contained in the Dividend; and alſo what remains after the Diviſor is taken out of the Dividend as oft as poſſible: So that in ſome Caſes there are four Numbers concerned in *Diviſion*; viz. the *Diviſor, Dividend, Quote,* and *Remainder.* *Examp.* 3 is contained in 14, 4 times, and 2 *remains.*

2. As *Multiplication* is only a compendious *Addition*; ſo is *Diviſion* only a compendious *Subtraction* of one Number out of another as oft as poſſible: For it is plain, that as oft as any Number is contained in another, ſo oft preciſely it can be taken out of it;

fo that if we find how oft the leffer can be taken out of the greater, we thereby find how oft it is contained in it : but to find how oft it can be taken out, is plainly the work of Subtraction; for by taking the leffer out of the greater, and the fame leffer out of the firft Remainder, and out of every Remainder fucceffively till the Remainder be 0, or lefs than the Subtractor, we have found what is required is *Divifion;* the Number of Subtractions being the Quote. So in thefe Examples, we find that 3 is contained in 12, 4 times; becaufe it can be taken out of it 4 times, and 0 remains. And 3 is contained in 14, 4 times, and 2 remains; becaufe being 4 times fubtracted, there remains 2.

Ex. 1.	Ex. 2.
12	14
3	3
9	11
3	3
6	8
3	3
3	5
3	3
0	2

But tho' the thing fought in *Divifion* may be found by Subtraction, yet it would be intolerable Labour in moft Cafes; and therefore the following Rules of *Divifion* are contrived, which do that by a few eafy fteps which cannot be done all at once, and would be too tedious to do by Subtraction.

Again *obferve,* That tho' there are here alfo, as in the preceding Operations, fome more fimple Cafes, the ready performance of which is ufeful in more complex Cafes : yet in moft Cafes we are left to guefs at the Anfwer in the feveral fteps of the Work ; with this help only, that we have a certain Rule for proving the Number gueffed to be right, and when it is wrong, how to come nearer to it at the next guefs, till at laft we find it; as you will prefently learn.

3. In all Cafes where there is a Remainder in the *Divifion,* or when the *Dividend* is not a Multiple of the *Divifor,* the Number called the *Quote* is the direct and proper Anfwer to that Queftion, How oft is the *Divifor* contained in the *Dividend?* yet the Remainder may be brought in fractionally as a Part of the *Quote,* making the Remainder the Numerator, and the *Divifor* the Denominator of a Fraction: And then we may fay, That the *Dividend* contains the *Divifor* fo many times as the *Integral Quote* expreffes, and fuch a Part or Parts of a time, (*i. e.* fuch a Part or Parts of the Divifor,) as that Fraction expreffes. Thus, for Example, 3 is contained in 14, 4 times, and 2 remains; which being $\frac{2}{3}$ Parts of 3, we may fay, that 14 contains 3, 4 times and $\frac{2}{3}$ Parts of a time; *i. e.* that it contains 4 times 3, and $\frac{2}{3}$ Parts of 3: And this mix'd Expreffion, $4 + \frac{2}{3}$; or thus, $4\frac{2}{3}$, may be called *The Complete Quote,* in diftinction from the Integral Quote. There will be the fame Reafon in all Cafes for completing the *Quote* by a Fraction made of the Remainder and Divifor, and underftanding it as we have done in this Example. Thus, univerfally, if any Number A is contained in another B, a number of times expreffed by D, with a Remainder R; then is the complete Quote $D\frac{R}{A}$: For any Number R being, the fame as R times 1, and 1 being fuch a Part of any Number A, as that Number denominates; *i. e.* the $\frac{1}{A}$ Part; therefore R is $\frac{R}{A}$ Parts of A. Wherefore, if by the words, *How oft,* in the Definition, we mean how many times and parts of a time, (as now explained,) the Integral Quote muft always have this Fraction, made of the Remainder and Divifor, annex'd to it as a part of the complete Anfwer of the Queftion.

Hence again *obferve,* That in this fenfe, the Dividend may be a leffer Number than the Divifor; for tho' a leffer Number does not contain a greater, yet it contains a certain Fraction of it, which is what we call containing it a certain part or parts of a time; and the Quote is a Fraction whofe Numerator is the Dividend, and its Denominator the Divifor. So 3 divided by 5, the *Quote* is $\frac{3}{5}$, fignifying that 3 contains 5, $\frac{3}{5}$ parts of a time; or that it contains $\frac{3}{5}$ parts of 5.

4. But again, for the fame *Reafon,* (*viz.* that any Number A is equal to or contains $\frac{A}{B}$ Parts of any Number B) the complete Quote of any Number divided by any other

Number

Number may be indefinitely expreſſed by a Fraction whoſe Numerator is the Dividend, and its Denominator the Diviſor. Thus 5 divided by 3, the *Quote* is $\frac{5}{3}$; and 3 divided by 5, the *Quote* is $\frac{3}{5}$. *Univerſally*, A divided by B, the *Quote* is $\frac{A}{B}$.

Now, as Fractions ariſe from *imperfect Diviſion* when the Dividend is not a Multiple of the Diviſor; ſo the Conſideration of Fractions does neceſſarily begin with *Diviſion of Whole Numbers:* For which Reaſon there are ſome things relating to Fractions muſt be explained in this Part; particularly I muſt here ſhew you, That the Fraction $\frac{A}{B}$ is in all Caſes equal in Value to the complete Quote of A divided by B; and this being demonſtrated, it will follow, that if we uſe ſuch an Expreſſion of a Quote in any Reaſoning or Operation, inſtead of the more direct and immediate Expreſſion of the Quote, which it is often very convenient to do, the Effect will be the ſame. And this I demonſtrate thus:

1. If A is leſſer than B, there is no other Quote. 2. If A is greater than B, then is $\frac{A}{B}$ an improper Fraction; and from the nature of ſuch a Fraction, (as it is defined and explained in *Chap.* 1.) this is manifeſt, That as oft as B is contained in A, ſo many Units (of that kind to which the Fraction refers) is the Fraction equal to, and to ſuch a proper Fraction more, whoſe Numerator is the Remainder, (after B is taken out of A as oft as poſſible) and its Denominator B. For when the Numerator and Denominator are equal, the Value of the Fraction is 1; ſo $\frac{5}{5}$ Parts, or $\frac{B}{B}$ Parts of any thing, is equal to that Thing or Unit: Therefore, if B is contained in A, R times without a Remainder, $\frac{A}{B}$ is equal to R times 1, or R. And if there is a Remainder *n*, the Value of it is $R + \frac{n}{B}$, or $R\frac{n}{B}$.

The *Diviſion* or *Quote* of two Numbers is alſo expreſſed by this Sign ÷ ſet betwixt them; the *Dividend* being ſet firſt, thus 5 ÷ 3, ſignifies 5 divided by 3. A ÷ B ſignifies A divided by B. It is alſo ſometimes expreſſed by this Sign), with the *Diviſor* before the *Dividend*, thus 3) 5. A) B.

Here then you have the *General Rule* of the LITERAL DIVISION.

COROLLARIES.

I. If the Integral Quote (or Number of times the whole Diviſor is contained in the Dividend,) is multiplied into the Diviſor, and to the Product be added the Remainder, (after the Diviſor is taken out of the Dividend as oft as poſſible) the Sum is equal to the Dividend. For Example, 3 is contained 4 times in 12, and nothing remains; therefore 4 times 3 is 12. Again, 3 is contained 4 times in 14, and 2 remains; therefore 4 times 3, and 2 added, is 14: For $4 \times 3 = 12$, and $12 + 2 = 14$. *Univerſally*, if $A \div B = q$, and nothing remaining, then is $Bq = A$. But if *r* remains, then is $Bq + r = A$.

II. The Remainder in *Diviſion* muſt always be leſs than the Diviſor; for elſe the Diviſor is not taken out of the Dividend as oft as poſſible: and this therefore is a Mark that the *Quote* is taken too ſmall: As again, if the Product of the Diviſor and Quote exceed the Dividend, it is a ſign the Quote is taken too great. And hence, laſtly, the Product of the Diviſor and integral Quote is the greateſt Multiple of the Diviſor contained in the Dividend.

SCHOLIUMS.

I. We have here learnt a mutual *Proof* of *Multiplication* and *Diviſion*, as theſe Operations are in their Effects directly oppoſite to one another. Thus, in *Multiplication*, if the Product is divided by any one of the Factors, the Quote is the other. And in *Diviſion*, if the Diviſor is multiplied by the integral Quote, and to the Product be added the Remainder, the Sum is the Dividend. Other *Proofs of Diviſion* you will find afterwards.

II. If

II. If we compleat the Quote by a Fraction made of the Remainder and Divifor, then it is a general Truth, that the Divifor multiplied by the compleat Quote, produces the Dividend: For being multiplied by the integral Quote, the Product is the greateft Multiple of the Divifor contained in the Dividend; and multiplied by the Fraction, (*i. e.* fuch a Fraction of it being taken,) it produces the Remainder; thus, $14 \div 3 = 4\frac{2}{3}$; then $4 \times 3 = 12$, and $\frac{2}{3}$ of $3 = 2$. And $12 + 2 = 14$. *Univerfally*, if $A \div B = q\frac{r}{B}$, then $A = Bq + \frac{r}{B}$ of $B = Bq + r$, as in *Corol.* 1. *Obferve* alfo, That as in Whole Numbers, it is no matter which of two Numbers is the Multiplier; fo, to multiply a Whole Number by a Fraction, or this by that, has the fame Effect, as will be explained in *Book* 2. But in the Cafe now before us, the Reafon is obvious. Thus to multiply B by $\frac{A}{B}$ is only taking $\frac{A}{B}$ Parts of B, which is A. Again, to multiply $\frac{A}{B}$ by B, is taking B times, A times $\frac{1}{B}$; (for $\frac{A}{B}$ is A times $\frac{1}{B}$.) But B times A = A times B, therefore B times A times $\frac{1}{B}$ = A times B times $\frac{1}{B}$ = A times 1; (for B times $\frac{1}{B} = 1$) or A. Wherefore to multiply the two Parts of the complete Quote by the Divifor, or to multiply the Divifor by thefe, the Sum of the Products is the Dividend. For $\frac{r}{B}$ of B = B times $\frac{r}{B}$ = r; and $Bq + r = A$. Or taking the complete Quote fractionally, then $\frac{A}{B}$ of B = B times $\frac{A}{B} = A$.

PROBLEM.
To Divide *one Number by another.*

CASE I. *When the* Divifor *is a Digit, or fingle Figure, and the* Dividend *either a Digit, or a Number of two Figures, whereof that in the place of Tens is lefs than the* Divifor.

Rule. Take fuch a Digit as, multiplied into the Divifor, will exactly produce the Dividend; but if there is not fuch a Digit, take the greateft, which multiplied into the Divifor, makes a Product lefs than the Dividend; that Digit is the integral Quote, and the Remainder (which muft be lefs than the Divifor) fet fractionally over the Divifor, compleats the Quote. The Reafon of this Rule is in *Schol.* 1. preceding.

Examp. 1. $12 \div 3 = 4$; becaufe $4 \times 3 = 12$. *Examp.* 2. $26 \div 4 = 6$, and 2 remains; fo the complete Quote is $6\frac{2}{4}$; for $6 \times 4 = 24$: then $24 + 2 = 26$. *Examp.* 3. $8 \div 5 = 1$, and 3 remains; fo the complete Quote is $1\frac{3}{5}$.

Whoever is familiar with the Table of *Multiplication*, can find at firft hearing the Anfwer to any Example of this Cafe. Or we may take help of that Table, thus: Seek the Dividend in the Table; and if it is not there, feek the greateft Number which is lefs than it; the Digit in the fame Line on the Side of the Table, or in the fame Column on the head of the Table, is the Divifor, and the other is the integral Quote.

CASE II. *To Divide any Number by another.*

Rule. Set the Divifor on the left of the Dividend, as in the following Examples; then take as many Figures from the Left-hand of the Dividend as are in Number equal to the Places of the Divifor; and if thefe Figures, confidered by themfelves, make a Number lefs than the Divifor, take in one Figure more, [which will neceffarily make a Number greater than the Divifor,] this we call the firft *Dividual*, (or partial *Dividend*.) We are then to find how oft the Divifor is contained in this Dividual; and here it is that we are left in a great meafure to guefs at the Figure fought: But this we know, that the Quote cannot exceed 9, as fhall be afterwards demonftrated. And then alfo (by *Corol.* 2. preceding,) it

muft

muft be fuch, that the Product of the Divifor by it do not exceed the Dividual, (for then it is too great a Figure,) and the Remainder (or Difference of the Dividual and Product) be lefs than the Divifor, (el e the Quote-Figure is too little;) and thus we muft find the Quote by trials. But to prevent too many ufelefs trials, we have this help, *viz.* Find, by *Cafe* I. how oft the firft Figure on the left is contained in the firft on the left of the Dividual, when this and the Divifor have equal number of Places; But in the firft two Figures on the left of the Dividual, when this has one place more than the Divifor; and this Number limits the gueffing, fo that you cannot take a greater; and if this happens to exceed 9, [which it will in no Cafe but the laft, and that where the firft of the two Figures in the Dividual is equal to the Divifor-Figure; for it is certain this will be found in thefe at leaft 10 times,] your guefs begins at 9. But then it will often happen that this Number is too great, and we have no other general Rule, or Help here, but to begin at this Figure and make trials, by multiplying the Divifor; for if the Product does not exceed the Dividual, that is the Figure fought: if it does exceed, take the next leffer Figure, and with it make the like trial; and go on fo gradually till you find a Figure whofe Product does not exceed the Dividual; for then the Remainder will certainly be lefs than the Divifor, which is the true Proof of the Quote-Figure's being right. Having thus found the firft Figure of the Quote, fet it down (on the Right-hand of the Dividend) and write the Product of it by the Divifor under the Dividual, and fubtract that out of this; and then before the Remainder (on the right) fet the next Figure of the Dividend, (or the Figure on the right of the firft Dividual,) and this Number is the fecond Dividual. Then in the fame manner as before, find how oft the Divifor is contained in this Dividual; fet the Figure found on the Right-hand of the Quote Figure laft found; then multiply the Divifor by it: write down the Product under the Dividual, and fubtract as before; then to the Remainder prefix the next Figure of the Dividend, and this is the next Dividual to be divided as before. In this manner proceed till every Figure of the Dividend is employed ftep by ftep: And *obferve,* that if any Remainder with one Figure of the Dividend prefix'd, makes a Number lefs than the Divifor, fet o in the Quote, and prefix alfo the next Figure; and fo on.

All the Quote Figures thus found, taken in order as they are placed as one Number, is the true Quote fought; and the laft Remainder is what the Dividend contains over fo many times the Divifor as the Quote expreffes.

I fhall next illuftrate this Rule by Examples.

Examp. I.

Div^r. Div^d. Quote.

4) 6392 (1598.

$$\begin{array}{r} 4 \\ \hline 23 \\ 20 \\ \hline 39 \\ 36 \\ \hline 32 \\ 32 \\ \hline 00 \\ \hline \end{array}$$

To divide 6392 by 4, I proceed thus: I feek how often the Divifor 4 is contained in 6, (the firft Figure of the Dividend) which is but once; therefore I fet 1 in the Quote; then $1 \times 4 = 4$ (or the Divifor multiplied by the Quote is 4,) which I write under the Dividual 6, and fubtracting, the Remainder is 2, to which I prefix 3, the next Figure of the Dividend, and then I take 23 for my next Dividual; and examining how oft 4 is contained in it, I find 5 times, (for had I taken 6, it were too great; for $6 \times 4 = 24$, which is greater than the Dividual 23; and had I taken 4, it were too little, for $4 \times 4 = 16$, and $23 - 16$, is 7, which is greater than the Divifor 4;) therefore I place 5 in the Quote on the Right of the former; and under 23 fet $20 = 5 \times 4$; then fubtracting, the Remainder is 3, to which I prefix the next Figure of the Dividend, *viz.* 9; then is 39 my next Dividual: and in this I find the Divifor 4 contained 9 times, which I write in the Quote on the right of the former; then $9 \times 4 = 36$, which I write under the Dividual 39, and the Remainder is 3, to which prefixing the next (and laft) Figure of the Dividend 2. This Number, *viz.*

32 is my next (and laft) Dividual, in which the Divifor is contained 8 times, which I fet in the Quote on the right of the preceding Figures; then fetting down the Product 32 ($= 8 \times 4$,) there is no Remainder; and the Quote fought is 1598, that is, 4 is contained in 6392, 1598 times.

Examp. 2.

```
36) 85609 (2378.
    72
    ---
    136
    108
    ---
    280
    252
    ---
    289
    288
    ---
      1 Remʳ.
```

To divide 85609 by 36, I proceed thus: There are two Places in the Divifor, therefore I take the firft two Places of the Dividend, *viz.* 85, which making a Number greater than the Divifor, I take them for the firft Dividual, and feeking how oft 3 (the firft Figure of the Divifor) is contained in 8 (the firft Figure of the Dividual, becaufe they have equal Places) I find it 2 times, which I find alfo by trial to be the true Quote of 85 by 36; fo I place 2 in the Quote, and fubfcribing the Product 72 ($= 2 \times 36$) I fubtract it from 85, the Remainder is 13, to which prefixing the next Figure of the Dividend 6, my next Dividual is 136. Then I feek how oft 3 (the firft Figure of the Divifor) is contained in 13, the firft two of the Dividual, (becaufe it has one place more than the Divifor,) and I find it 4 times: but this is too great for the whole Divifor, (becaufe $4 \times 36 = 144$) therefore I try the next Figure 3, and find it right; therefore I fet 3 in the Quote, and fubfcribing the Product 108 ($= 3 \times 36$,) I fubtract it from 136, and the Remainder is 28; to which prefixing 0, the next Figure of the Dividend, my next Dividual is 280. Then I feek how oft 3 is contained in 28, and I find 9 times; but this is too great, (for $9 \times 36 = 324$.) I take again 8, and find it alfo too great, (for $8 \times 36 = 288$) and at laft I find 7 to be right; therefore I fet 7 in the Quote, and fubfcribing the Product 252 ($= 7 \times 36$) the Remainder is 28; to which prefixing 9, the next and laft Figure of the Dividend, I have for my next and laft Dividual 289, in which I find as before, that the Divifor is contained 8 times, and 1 remains. So the true Quote is 2378, and 1 over; which is $2378\frac{1}{36}$.

Examp. 3.

```
465) 2744897 (5903
     2325
     ----
     4198
     4185
     ----
      1397
      1395
      ----
        2 Remʳ.
```

To divide 2744897 by 465, I proceed thus: my firft Dividual is 2744; I feek how oft 4 (the firft Figure of the Divifor) is contained in 27 (the firft two of the Dividual.) I find it 6; but this is too great, and I take 5, and find it right. Then multiplying and fubtracting, and prefixing to the Remainder the next Figure of the Dividend, my next Dividual is 4198; and here I find the Divifor contained 9 times. Then proceeding as before, my next Dividual is 139; which being lefs than the Divifor, I fet 0 in the Quote, and then prefix another Figure; fo that my next Dividual is 1397, in which the Divifor is contained 3 times, and the Remainder is 2; fo the true Quote is 5903; or, taking in the Remainder, it is $5903 \frac{2}{465}$.

To divide 3235386 by 462. The firft Dividual is 3235, in which the Divifor is contained 7 times, and the Remainder is 1; the next Dividual is 13, in which the Divifor is not contained; therefore I fet 0 in the Quote, and bringing down the next Figure, I have 138 for a new Dividual, which is alfo lefs than the Divifor, therefore I fet another 0 in the Quote; and bringing down another Figure, the next Dividual is 1386, in which the Divifor is contained 3 times, and 0 remains.

Examp. 4.

```
462) 3235386 (7003
     2234
     ----
      1386
      1386
      ----
       0000
```

Examp. 5.

```
372) 149100 (400
     1488
     ----
       300 Remʳ.
```

To divide 149100 by 372. The firft Dividual is 1491, and the Quote of this is 4, then the Remainder is 3, and the reft of the Dividuals are 30 and 300, which are each lefs than the Divifor, and therefore the Quote is 400.

I

Divide

Examp. 6.

$$62)\ 210800\ (3400$$
$$186$$
$$248$$
$$248$$

To Divide 210800 by 62, the Quote is 3400.

In Cafes like this, when there is no Remainder in any ftep, and that the Remaining Figures of the Dividend are all o's, we have no more to do than to join as many o's to the preceding Figure, found in the Quote.

Thefe Examples will be fufficient to make a diligent Learner underftand this Rule; and I fhall only fet down a few more Examples, with their Anfwers, leaving the Operation for an Exercife to the Student.

Examp. 7. To divide 5713070046 by 678, the Quote is 8426357.
Examp. 8. To divide 6069944827 by 8376, the Quote is $724683 \frac{19}{8376}$.
Examp. 9. To divide 293682135936 by 8405, the Quote is $34960078 \frac{346}{8405}$.

In order to the *Demonftration* of the preceding Rule, and for the fake of fome other fpecial Rules to follow, we muft premife the following Truths in the Theory of *Divifion*.

L E M M A I.

1. If two Numbers confift of an equal Number of Places, the leffer is not contained in the greater above 9 times.

2. Again, If the greater of two Numbers has but one Place more than the leffer; and fuppofing alfo that, excluding the firft Figure on the Right of that greater Number, the remaining Figures on the Left make a Number lefs than the leffer given Number, then this leffer Number is not contained in the greater above 9 times.

DEMON. *Part* 1. If a Cypher is prefix'd to the leffer of two Numbers, (which have both the fame Number of Places,) it is thereby multiplied by 10; and confequently that is the leaft Number which contains it 10 times: but the other given Number having one Place fewer than this Product, is a leffer Number, and confequently does not contain the leffer given Number 10 times, or does not contain it above 9 times. *Example*, 11 is lefs than 99; but is not contained in it 10 times, for 10 times 11 is 110, which is greater than 99. *Univerfally*, Let A be the greater, and B the leffer of two Numbers having an equal Number of Places; B × 10 contains B precifely 10 times, and it is a Number that has one Place more than B or A, and confequently is a greater Number than A; wherefore B is not contained 10 times in A.

Part 2. The leffer given Number (as 476) is greater, by fuppofition, than as many Places (475) on the left of the greater given Number (4759;) and muft exceed it by at leaft 1: therefore 10 times this leffer Number, (viz. 4750,) muft want at leaft 10 of 10 times the given leffer Number, (viz. 4760.) But whatever Digit we add to this deficient Product, or put in the Place of the 0, (making in the prefent *Examp.* 4759,) it cannot make up the defect of 10; and therefore the given leffer Number (476) is not contained in the given greater Number (4759) 10 times, or not above 9 times.

SCHOL. The fecond Part of this *Lemma* is but a particular Cafe (accommodated to our prefent purpofe) of a more general *Theorem*; which is this, viz. If any Number A, is greater than another, B; and if B is multiplied by any Number R, then A is not contained R times in R B, nor yet in the Sum of R B, and any Number N which is lefs than R; *i. e.* in R B + N.

The *Reafon* is; fince A is not once contained in B, neither is R A once contained in R B, which muft want at leaft as many Units as R to make it equal to R A; and fince N is lefs than R, R B + N cannot be equal to R A; *i. e.* A, which is contained precifely R times in R A, is not contained R times in R B + N, which is lefs than R A.

2

L E M-

L E M M A II.

If any Number N is refolved into any Parts A, B, C, *&c.* *i. e.* if N = A + B + C, *&c.* then,

1. If all thefe Parts A, B, C, *&c.* are feverally Multiples of any Number D, or all except one; then dividing A, B, C, *&c.* feverally by D, the Sum of the Quotes is equal to the Quote of N ÷ D. (*Examp.* 1, 2, 3.) And the Remainder in the Divifion of that Part which is not a Multiple of D, is the Remainder in the Divifion of N ÷ D.

2. If there are more than one of the Parts of N, that are not Multiples of D; and if the Sum of the Remainders, in the Divifion of thefe Parts that are not Multiples of D, is lefs than D, then the Sum of the Integral Quotes, is the Integral Quote of N ÷ D; and the Sum of the Remainders, is the Remainder in the Divifion of N ÷ D. (*Examp.* 4.)

3. If the Sum of the Remainders is equal to, or greater than D; then, being divided by D, and the Integral Quote added to the Sum of the Integral Quotes of the Parts of N, this laft Sum is the Integral Quote of N ÷ D; and the Remainder in the Divifion of the Sum of the Remainders, is equal to the Remainder in the Divifion of N ÷ D. (*Examp.* 5.)

DEMON. The Reafon of all thefe Articles is eafily feen from the Equality of the Whole and all its Parts. In the firft and fecond Article it is obvious: for as oft as D is contained in A, B, C, *&c.* feverally, fo oft at leaft as the Sum of thefe times, it muft be contained in the whole N; and if the Sum of the Remainders in the Divifion of the Parts A, B, C, *&c.* is lefs than D, then it is plain that D is contained no oftner in N than the Sum of the times it is contained in all its Parts A, B, C, *&c.* and the Remainder in N ÷ D, muft be the Sum of the Remainders in the Divifion of the Parts, when this Sum is lefs than D; but if this Sum is equal to, or exceeds D, (as fuppofed in Article 3.) then fince thefe Remainders are Parts of the Dividend, it is evident, that as oft as D is contained in their Sum, that muft be added to the Sum of the times it is contained in A, B, C, *&c.* and this laft Sum is the times it is contained in N; and the Remainder on the Divifion of the Sum of the Remainders, is the Remainder in dividing N by D.

Examp. 1.
D N = A + B + C.
Divifor 4) 36 = 16 + 12 + 8. Dividends.
Quotes 9 = 4 + 3 + 2.

Examp. 2.
Divifor 5) 48 = 25 + 15 + 8. Dividends.
Quotes 9 = 5 + 3 + 1.
Remainders 3 = 3.

Examp. 3.
Divifor 6) 46 = 24 + 18 + 4. Dividends.
Quotes 7 = 4 + 3 + 0.
Remainders 4 = 4.

Examp. 4.
Divifor 6) 53 = 20 + 27 + 6. Dividends.
Quotes 8 = 3 + 4 + 1.
Remainders 5 = 2 + 3.

SCHOL. If we take the complete Quotes by Fractions made of the Remainders and Divifor; then it is an *Univerfal Truth*, That the Sum of the Quotes of the Parts of N divided feverally by any Number D, is equal to the Quote of N divided by D: For the fractional Parts of the Quotes have all the fame Denominator D, and their Numerators are the feveral Remainders. But from the nature of Fractions it is obvious, that feveral Fractions having the fame Denominator, and being referred to the fame Integer, their Sum is the Sum of the Numerators, applied as a Numerator to the fame Denominator. *Examp.* $\frac{2}{7}$ of any thing, and $\frac{3}{7}$ of the fame thing, make $\frac{5}{7}$. So that if the Sum is an improper Fraction, we find its Value in a whole Number, or with a proper Fraction annex'd, by dividing the Numerator

Divisor 6) 88 = 46 + 29 + 13. Divid.
Quotes   14 , 7 $\frac{1}{4}$ + 4 $\frac{1}{4}$ + 2.
Remainders 4 , 4 + 5 + 1.

In this laſt Example, the Sum of the Remainders is 4 + 5 + 1 = 10, which is greater than the Diviſor 6; and being divided by it, the Quote is 10 ÷ 6 = 1, and 4 remains. Then this Quote 1 added to the Sum of the former Quotes, the Sum is 7 $\frac{1}{4}$ + 4 $\frac{1}{4}$ + 2 $\frac{1}{4}$ + 1 = 14, the Quote of 88 ÷ 6, and the Remainder of 88 ÷ 6, is the ſame as that of 10 ÷ 6, which is 4.

merator by the Denominator, (as before explained.) It is plain then, that if the Fractions belonging to the complete Quotes of the Parts of N divided by D, are added together, and the Value of the Sum added to the Integral Quotes, the Operation is the ſame as expreſſed in the *Lemma*: for it is adding the Remainders; and if their Sum exceeds the Diviſor, taking the Number of times the Diviſor is contained in it, and adding this Quote to the Sum of the Integral Quotes; which makes the Univerſal Truth here propoſed evident.

COROLLARIES.

I. If a Number N is reſolved into any Number of Parts, and theſe Parts be divided ſeverally by any Number D, in this manner, *viz.* Firſt divide one Part, and if there is a Remainder, add it to another Part, and divide the Sum; and ſo on, adding the Remainder of every Diviſion to the next Part; and if any Part with the preceding Remainder is leſs than the Diviſor, then we add another. Having thus gone through all the Parts, the Sum of the Quotes is the Quote of the firſt Number N divided by the ſame Diviſor D; and the laſt Remainder in the Diviſion of the Parts, is the Remainder in the Diviſion of N by D.

That this is in Effect the ſame Caſe as the firſt Article of the preceding *Lemma*, or a plain Conſequence of it, will be obvious by conſidering, that if the firſt Remainder is taken out of the firſt Dividend, it leaves a Multiple of the Diviſor, *viz.* the Product of the Diviſor and Quote; and the ſame being true in all the reſt of the Steps, it follows that the Caſe is the ſame as if N were reſolved into Parts equal to theſe Multiples of the ſeveral Quotes by the Diviſor; all which Multiples with the laſt Remainder make up the Dividend N. For Example, 50 = 17 + 8 + 25, then 17 ÷ 3 = 5, and 2 remains, which added to 8, makes 10; then 10 ÷ 3 = 3, and 1 remains, which added to 25, makes 26; and 26 ÷ 3 = 8, and 2 remains; laſtly, the Sum of the Quotes is 5 + 3 + 8 = 16, the Integral Quote of 50 ÷ 3; and the laſt Remainder 2, is the Remainder of 50 ÷ 3. And this Work is the ſame in effect as if we reſolve 50 into theſe Parts, 15 (= 3 × 5) + 9 (= 3 × 3) + 24 (= 3 × 8.)

Or this Truth is plain independently of the *Lemma*, becauſe the Diviſor is taken out of every Part of the Dividend as oft as poſſible, by carrying the Remainder of every Part forward to the next.

II. If the ſame Diviſor D is applied to two different Dividends, whereof the greater is a Multiple of the leſſer, as N and Nm; then if N contains D, a number of times q, without a Remainder; Nm will contain D, m times as oft as N does, or mq times; *i. e.* if N ÷ D = q, then Nm ÷ D = mq. Again, if N ÷ D has a Remainder r, then Nm will contain D at leaſt mq times with a Remainder equal to mr. And if mr is equal to, or greater than D, then, as oft as D is contained in mr, ſo many times oftner than mq is it contained in Nm. The Deduction of this from the *Lemma* is plain; becauſe Nm is reſolvable into N + N + N, *&c.* taking N as oft as m expreſſes; ſo that if N ÷ D = q, and r remains; then D is contained in Nm at leaſt mq times, with a Remainder equal to rm. See theſe Examples, wherein $m = 100$.

Examp. 1. $\begin{array}{c} \text{D N} \quad q. \\ 3)\,18\,(6. \end{array}$ *Examp.* 2. $\begin{array}{c} \text{D N} \quad q. \\ 3)\,23\,(\,7. \\ \underline{21} = \text{D}\,q. \\ \underline{2} = r. \end{array}$ then $\begin{array}{c} \text{D N}\,m \quad mq. \\ 3)\,2300\,(700. \\ \underline{2100} = \text{D} \times m\,q. \\ \underline{200} = r\,m. \end{array}$

$\begin{array}{c} \text{N}\,m \quad m\,q. \\ 3)\,1800\,(600. \end{array}$

SCHOL. Thefe Examples are of the Kind which we have particular Ufe for in De-monftrating the *Rule of Divifion:* And we have this further to be carefully remarked in all Examples like the fecond, *viz.* That tho' the Remainder ($m\,r$) in the fecond Part of the Example, is greater than the Divifor, whereby the Integral Quote is not fo great as the number of times that D may be got in N m, yet of the Number it wants to be added to it, (which is the Quote of $m\,r$ by D,) all the Figures will fall in the Places of the o's ftanding on the right of that Part of the Quote which is already found, and can never be of the fame local Value with any of the other Figures. So 3 is found in 2300 as many times oftner than 700 times, as the Quote of 200 by 3; yet no Figure of this Quote can rife to the Place of 100; the Reafon of which, and of all fuch Cafes, is explained in *Schol.* to *Lemma* I. For 3 being greater than 2, is not contained 100 times in 200. And which will alfo be true, tho' we fet any other Figure in the Place of the o's that ftand on the right of the Remainder, fince the Remainder without thefe o's is lefs than the Divifor. See the *Scholium* referred to.

III. If N($=$A$+$B) is a Multiple of C; and if A is alfo a Multiple of C, then fo muft B be. Again, if N$=$A$+$B$+$C$+$D, *&c.* and if N is a Multiple of R; and alfo if each of the Parts of N to the laft, are Multiples of R, fo muft that laft be.

LEMMA III.

If one Number is divided by another, and the Quote again divided by the fame, or any other, and every fucceeding Quote again divided as long as you pleafe or can; the laft Quote will be equal to the Quote of the firft Dividend by the continual Product of all thefe Divifors.

DEMON. 1. We fhall firft fuppofe the feveral Dividends are Multiples of the Divifors; and in this Cafe it will eafily appear, as in the annex'd Example.

$\begin{array}{l} 2)\,168\,(84. \\ 3)\,84\,(28. \\ 7)\,28\,(4. \\ \text{then becaufe} \\ 2\times3\times7=42; \\ \text{therefore} \\ 42)\,168\,(4. \end{array}$

The Reafon is this. If we take the laft Quote and all the Divifors in a reverfe Order, and multiply them continually, they muft produce the firft Dividend, (by what is already fhewn of the mutual Proof of *Multiplication* and *Divifion*.) Thus, $4\times7=28.$ $28\times3=84.$ $84\times2=168.$ But we may take thefe Factors in any Order, they will pro-duce at laft the fame Number, (by *Lemma* II. in *Multiplication*.) And if we order them fo as the laft Quote in Divifion be the laft Factor in multiplying, the Truth propofed will be manifeft, thus; Becaufe $4\times7\times3\times2=168.$ Therefore alfo $2\times3\times7\times4=168.$ But $2\times3\times7=42;$ therefore $42\times4=168;$ and $168\div42=4.$ The fame Reafoning will hold in all Cafes, which we may reprefent by Univerfal Characters. Thus; if $A\div b=M;$ and $M\div c=N;$ and $N\div d=q:$ then $A\div bcd=q.$ For, $qd=N;$ $Nc=M;$ and M $b=A;$ that is, $qdcb=A;$ or $bcdq=A:$ But $bcdq\div bcd=q;$ *i.e.* $A\div bcd=q.$

2. Suppofe there is a Remainder in each Divifion, yet the laft Quote will ftill be equal to the Quote of the firft Dividend by the Product of all the fingle Divifors, tho' the Re-mainder will not be the fame as the Remainder of the laft Divifion. That we may fee the Truth propofed in this Cafe, and alfo how to find by the feveral Divifors and Remain-

ders

ders, what the Remainder would be upon dividing by the Product of the Divisors, we shall confider the *Example* annex'd; wherein the thing propofed is proved. But to fee the Reafon of it, we muft take the *reverfe Multiplication*, as in the Margin on the Right.

$$3)\ 4^-9\ (159$$
rem. 2

$$5)\ 159\ (31$$
rem. 4

$$4)\ \ 31\ (7$$
rem. 3

then becaufe

$$3 \times 5 \times 4 = 60$$
therefore

$$60)\ 479\ (7$$
rem. 59

Thus; Take the laft Quote and the feveral Divifors, as fo many fimple *Factors*, and multiply them continually, taking in the correfpondent Remainders with the Product, (to make up the feveral Dividends.) Now the firft thing to be fhewn from this Multiplication is the reafon why, That tho' the laft Product (479) exceeds the continual Product of the *Factors* $7 \times 4 \times 5 \times 3$ $= 420$, (as it muft do, becaufe of the Numbers taken in;) yet it cannot exceed it by a Number as great

laft quote	7 :	3, laft rem.
laft div^r.	4	
2^d quote	31 :	4, 2^d rem.
2^d div^r.	5	
1ft quote	159 :	2, 1ft rem.
1ft div^r.	3	
	479	

as the continual Product of the feveral *Factors* excluding the firft 7, (*i. e.* the feveral Divifors, *viz.* $4 \times 5 \times 3 = 60$;) which is thus fhewn.

The Number by which the laft Product, or Sum (479) exceeds the continual Product of all the fimple *Factors* (420,) is plainly to be found thus; *viz.* Take the Product of the laft Remainder (3) by the laft Divifor but one, (5;) then to this Product (15) add the next preceding Remainder (4,) and multiply the Sum (19) by the next preceding Divifor (3,) and to this Product (57) add the next preceding Remainder (2,) and fo on: for the laft Remainder 3 is taken in with the firft Multiplication; then it is multiply'd in the fecond Multiplication by 5, with which the fecond Remainder is taken in, (which makes $15 + 4 = 19$.) Then is all this multiply'd in the third Multiplication by 3, and the firft Remainder 2 is taken in, and the whole is 59; fo the true Remainder fought is 59. But this muft be always lefs than the Product of the Divifors, becaufe in the feveral Multiplications the Numbers taken in are lefs than the correfpondent Multipliers, (for they are Remainders of a Divifion wherein that Multiplier was the Divifor;) wherefore the Product of any of the Divifors by the following Remainder, (or Remainder of the next Divifion,) with the prefent Remainder added, is lefs than the Product of the fame Divifor by the following Divifor, and confequently the continual Product of all the Divifors is greater than the Product of all the Divifors (excluding the laft,) with the Remainders taken as directed. This being once clear, the Truth concerning the laft Quote is manifeft; for nothing elfe can make it different from the laft Cafe but thefe Remainders, or Numbers added in the Multiplication, which can never make the Quote lefs: and, by what is now fhewn, they cannot make it greater, becaufe all the Increafe upon the laft Dividend is lefs than the total Divifor, (or Product of the particular Divifors.)

As to the fractional Part of the Quote, it will be of the fame Value, tho' not of the fame Expreffion. But the Truth of this will not appear fo eafily till we have learnt the *Doctrine of Fractions*, where you'll find it particularly explained.

DEMONSTRATION *of the preceding Rule of* CASE II. *of* DIVISION.

As to finding every one of the Figures of the Quote ſingly, as the true Quotes of the Diviſor into the ſeveral Dividuals conſidered by themſelves, we need no further Demonſtration; becauſe each of them is found by trial, and proved to be true by a certain and infallible Mark, (diſcovered in *Corol.* 2. to the *Definition*) before it is admitted. All that I have more to add, is to obſerve, That you have in *Lemma* 1. the *Reaſon* that the *Diviſor* can never be found in the *Dividual*, (or *Partial Dividend*, as it is to be taken by the Rule) above 9 times.

The only thing then that remains to be proved is, That the ſeveral Figures of the Quote taken as one Number, according to the Order in which they are placed, is the true Quote of the total *Dividend* by the *Diviſor*; and this will be eaſily ſhewn, thus: It is plain, that in this Operation we have reſolved the *Dividend* into Parts, and divided them ſeverally in the manner propoſed in *Corol.* 1. *Lem.* 2. For we have firſt taken one Dividual, then added the Remainder of that to another Part of the *Dividend*, and after dividing this, we have added to the Remainder another Part of the *Dividend*, and ſo on. But we have conſidered the ſeveral Dividuals without regard to the Places they poſſeſs in the *Whole Dividend*, and thereby taken them in a leſs than their true Value; but if that Defect is ſupplied by placing the ſeveral Quotes, (or Parts of the total Quote) ſo as they have the true Value they ought to have from the complete Value of their reſpective Dividuals, (or Parts of the *Dividend*,) and that by ſo placing them they make one Number equal to the Sum of their complete Values; then is that Number the true Quote ſought, (by the ſaid *Corol.* 1. *Lem.* 2.) But thus it actually is: for the complete Value of the firſt Dividual is 10, or 100, &c. times the Value in which it is taken in the Operation, according as there is one, two, &c. Figures ſtanding before it: Alſo its quote Figure ſtanding firſt on the Left-hand, there are as many Figures of other Quotes ſet before it, as the Number of remaining Figures in the *Dividend*; becauſe for each of theſe there is, by the Rule, a Figure put in the Quote: therefore this firſt quote Figure receives by its Place, a Value 10, or 100, or 1000, &c. times its ſimple Value, according as there are one, or two, or three, &c. Figures before it, correſponding to the true Value of the Dividuals, (as ought to be done by *Corol.* 2. *Lem.* 2.) Therefore this firſt quote Figure taken in its complete Value from the place it ſtands in, is the true Quote of the Diviſor in the complete Value of the firſt Dividual. For the ſame reaſon, all the reſt of the Figures in the Quote taken according to their Places, are each the true Quote of the Diviſor in the complete Value of their Dividuals; becauſe as the firſt Figure on the right of each ſucceeding Dividual, is one Place more to the right of the preceding, (or has one Figure fewer ſtanding before it,) ſo ought their Quotes to have; and ſo are they alſo ordered: Conſequently taking all the quote Figures in order as they are placed by the Rule, they make one Number, which is equal to the Sum of the true Quotes of the ſeveral Parts of the *Dividend*; which is therefore the true Quote of that whole *Dividend*.

To leave no Obſcurity in this Demonſtration, I ſhall illuſtrate it by two Examples. In which I ſhall ſet down the Dividuals, Quotes, and Remainders according to their true Values.

Examp. 1.

Divifor. Dividend.

```
         36 ) 85609   Quotes.
1ft dividual  85000 ( 2000
36 × 2000 = 72000
1ft remain.   13000
       add     600
2d dividual   13600 (  300
36 × 300 = 10800
2d remain.     2800
       add      00
3d dividual    2800 (   70
36 × 70 =      2520
3d remain.      280
       add        9
4th dividual    289 (    8
36 ×  8 =       288    2378 Sum
laft remain.      1   of the Quotes.
```

In the firft Example, the Dividend 85609 is refolved into thefe Parts, *viz.* 8500 + 600 + 00 + 9. For tho' the firft Dividual is confidered as 85, yet it is truly 85000; and therefore its Quote inftead of 2, is 2000, and the Remainder 13000; and fo of the reft, as you fee in the Operation. But if we take the Multiples of the Divifor by the feveral Quotes, with the laft Remainder, and confider the Dividend as diftributed into thefe Parts, (which are here 72000 + 10800 + 2520 + 288 + 1,) then the Work is reduced to the Conditions of *Lemma* 2. *Article* 2.

Examp. 2.

Divifor. Dividend.

```
        465 ) 2744897  Quotes.
1ft dividual  2744000 ( 5000
465 × 5000 = 2325000
1ft rem.       419000
       add        800
2d dividual    419800 (  900
465 × 900 =    418500
2d rem.          1300
       add         97
3d dividual      1397 (    3
465 ×   7 =      1395    5903
laft rem.           2
```

In this fecond Example, when we have got the fecond Quote, the Remainder is 1300; then we add the two next Figures of the Dividend, becaufe the Figure of the Quote muft be of the fame local Value as the laft of thefe Figures: For fince 465 is not contained in 139, it is not contained 10 times in 1390; and fo the next Figure in the Quote after 9 muft be 0, and the fignificant Figure of the Quote of 1390 ÷ 465, muft be in the fecond Place after 9, *i. e.* in the prefent Example, in the Place of Units: and therefore we take in alfo the Figure 7, which is in the Place of Units, to find at once all that Part of the Quote which belongs to the Place of Units; for had we divided 13900 by 465, the Quote is 2, and the Remainder 460; to which adding the laft Figure of the Dividend 7, the Sum is 467, in which the Divifor is contained once, and 2 remains; and fo thefe two Quotes both in the place of Units, *viz.* 2 + 1 make 3, which is more conveniently found, as in the Operation in the Margin. The like reafon you'll find in all Cafes where there are 0's in the Quote. And for the laft Example, take 113764 ÷ 28 = 4063, the firft Dividual is 113000, the Quote 4000, and the Remainder 1000; to which if we add 700, the Sum 1700 does not contain the Divifor 28 fuch a number of times as can fall in the Place of Hundreds; therefore we take in another Figure, which makes 1760; and the Quote 6 falls in the fecond Place after the preceding Quote 4. The laft Figure of the Quote is 3.

§. II. PARTICULAR RULES *for contracting the Work of Division in certain Cafes: And, for managing it with more Certainty, tho' with more Work, in all Cafes.*

CASE 1. When the Divifor is a Digit, the multiplying of the Divifor and Quotes, and alfo the Subtraction of the Products from the Dividuals, may be eafily performed without writing down any thing but the Remainders; and thefe, with the Quotes, fet more conveniently, as in this *Example*; wherein 37546 is divided by 4. Thus, 4 in 37 is 9 times, and 1 remains; the Quote 9 I fet under the Dividend, and the remainder 1 above; then this Remainder, with the next Figure of the Dividend prefix'd, makes 15; the next Dividual, in which 4 is contained 3 times, and 3 remains; the Quote 3 is fet after the preceding Quote, and the Remainder 3 over the 5 of the Dividend. Then is the next Dividual 34, whofe Quote is 8, and 2 remains; then the laft Dividual is 26, and the correfponding Quote is 6, and 2 remains. *Again*, in this *Cafe* it will be very eafy to do the Work without writing down the Remainders, only conceiving them, in the places where they ought to be. And the Conveniency of doing it this way, you'll fee in *Cafe* 3. Obferve alfo, that you may eafily ufe the fame Practice, if the Divifor is 11 or 12.

$$\begin{array}{l} 1322 \\ 4)37546 \\ 9386\tfrac{1}{4} \end{array}$$

CASE 2. If the Divifor has o's in the firft Places next the Right-hand, take no notice of them in the Operation, making the Divifor only the remaining Figures on the Left; and exclude as many Figures, whatever they are, from the Right of the Dividend, as thofe o's of the Divifor; making the remaining Figures on the Left the Dividend. Having finifhed this Divifion, you have found the integral Quote fought: And for the fractional Part, to the Remainder of the Divifion prefix all the Figures excluded from the Dividend; that is the true Remainder that would happen if the Divifion were done by the common Method: This Remainder, with the given Divifor, makes the Fraction. But obferve, that if there are any o's ftanding clear on the Right-hand of this Remainder, you may omit them all (in making your Fraction) if they do not exceed the Number of o's excluded from the Divifor; or as many of them, as are equal in Number to thofe in the Divifor; and omitting the fame Number of o's in the Divifor, of the remaining Figures on the Left make your Fraction.

Example 1. To divide 84700 by 4600, I divide 847 by 46, the Quote is 18, and 19 remains; but the true Remainder is 1900: Making this Fraction $\frac{1900}{4600}$, which is equivalent to this $\frac{19}{46}$.

Ex. 2. To divide 3640 by 800, I divide 364 by 8, the Quote is 45, and 4 remains; but the true Remainder is 420: Making this Fraction $\frac{420}{800}$, equal to this $\frac{42}{80}$.

Examp. 3. To divide 68704 by 2400, I divide 687 by 24; the Quote is 28, and 15 remains; but the true Remainder in the Queftion is, 1504, and the Fraction is $\frac{1504}{2400}$.

Examp. 4. To divide 367854 by 800, I divide 3678 by 8; the Quote is 459, and the Remainder 6; but the true Remainder is 654, and the Fraction $\frac{654}{800}$.

The *Reafon* of this Rule is contained in *Lem.* 4. Thus, if an equal Number of o's are excluded from Divifor and Dividend, the remaining Figures on the left exprefs like aliquot Parts of them, *viz*, a tenth Part if one o; a hundredth, if two o's were excluded, and fo on. But like aliquot Parts of two Numbers make the fame Quote as their Wholes, (by *Lem.* 4.) *Ex.* 46 and 847 are the hundredth Parts of 4600 and 84700, and fo have the fame Quote 18, with a Remainder 19, which is the hundredth Part of the Remainder in dividing 84700 by 4600; and therefore the two o's cut off from the Dividend are to be prefix'd to it, to give it the true Value. *Again*, if the Figures excluded the Dividend are not all o's, yet if we fuppofe them fo, the Quote is right: And that it cannot be increafed by the Value of the Figures cut off, whatever they are, is plain; be-

K caufe

Examp. 1.

```
          Divifor.  Dividend.
               36 ) 85609   Quotes.
1ft dividual   85000 ( 2000
36 × 2000 = 72000
1ft remain.    13000
       add       600
2d dividual    13600 (  300
  36 × 300 = 10800
2d remain.      2800
       add        00
3d dividual     2800 (   70
  36 × 70 = 2520
3d remain.       280
       add         9
4th dividual     289 (    8
  36 × 8 =        288      2378 Sum
laft remain.       1      of the Quotes.
```

In the firft Example, the Dividend 85609 is refolved into thefe Parts, *viz.* 8500 + 600 + 00 + 9. For tho' the firft Dividual is confidered as 85, yet it is truly 85000; and therefore its Quote inftead of 2, is 2000, and the Remainder 13000; and fo of the reft, as you fee in the Operation. But if we take the Multiples of the Divifor by the feveral Quotes, with the laft Remainder, and confider the Dividend as diftributed into thefe Parts, (which are here 72000 + 10800 + 2520 + 288 + 1,) then the Work is reduced to the Conditions of *Lemma* 2. *Article* 2.

Examp. 2.

```
          Divifor.  Dividend.
             465 ) 2744897   Quotes.
1ft dividual   2744000 ( 5000
465 × 5000 = 2325000
1ft rem.       419000
       add        800
2d dividual    419800 (  900
465 × 900 = 418500
2d rem.          1300
       add         97
3d dividual      1397 (    3
465 × 7 =        1395     5903
laft rem.           2
```

In this fecond Example, when we have got the fecond Quote, the Remainder is 1300; then we add the two next Figures of the Dividend, becaufe the Figure of the Quote muft be of the fame local Value as the laft of thefe Figures: For fince 465 is not contained in 139, it is not contained 10 times in 1390; and fo the next Figure in the Quote after 9 muft be o, and the fignificant Figure of the Quote of 1390 ÷ 465, muft be in the fecond Place after 9, *i. e.* in the prefent Example, in the Place of Units: and therefore we take in alfo the Figure 7, which is in the Place of Units, to find at once all that Part of the Quote which belongs to the Place of Units; for had we divided 13900 by 465, the Quote is 2, and the Remainder 460; to which adding the laft Figure of the Dividend 7, the Sum is 467, in which the Divifor is contained once, and 2 remains; and fo thefe two Quotes both in the place of Units, *viz.* 2 + 1 make 3, which is more conveniently found, as in the Operation in the Margin. The like reafon you'll find in all Cafes where there are o's in the Quote. And for the laft Example, take 113764 ÷ 28 = 4063, the firft Dividual is 113000, the Quote 4000, and the Remainder 1000; to which if we add 700, the Sum 1700 does not contain the Divifor 28 fuch a number of times as can fall in the Place of Hundreds; therefore we take in another Figure, which makes 1760; and the Quote 6 falls in the fecond Place after the preceding Quote 4. The laft Figure of the Quote is 3.

§. II. PARTICULAR RULES *for contracting the Work of Divifion in certain Cafes: And, for managing it with more Certainty, tho' with more Work, in all Cafes.*

CASE 1. When the Divifor is a Digit, the multiplying of the Divifor and Quotes, and alfo the Subtraction of the Products from the Dividuals, may be eafily performed without writing down any thing but the Remainders; and thefe, with the Quotes, fet more conveniently, as in this *Example;* wherein 37546 is divided by 4. Thus, 4 in 37 is 9 times, and 1 remains; the Quote 9 I fet under the Dividend, and the remainder 1 above; then this Remainder, with the next Figure of the Dividend prefix'd, makes 15; the next Dividual, in which 4 is contained 3 times, and 3 remains; the Quote 3 is fet after the preceding Quote, and the Remainder 3 over the 5 of the Dividend. Then is the next Dividual 34, whofe Quote is 8, and 2 remains; then the laft Dividual is 26, and the correfponding Quote is 6, and 2 remains. *Again,* in this *Cafe* it will be very eafy to do the Work without writing down the Remainders, only conceiving them, in the places where they ought to be. And the Conveniency of doing it this way, you'll fee in *Cafe* 3. Obferve alfo, that you may eafily ufe the fame Practice, if the Divifor is 11 or 12.

$$\begin{array}{r} 1322 \\ 4)37546 \\ \hline 9386\tfrac{1}{4} \end{array}$$

CASE 2. If the Divifor has o's in the firft Places next the Right-hand, take no notice of them in the Operation, making the Divifor only the remaining Figures on the Left; and exclude as many Figures, whatever they are, from the Right of the Dividend, as thofe o's of the Divifor; making the remaining Figures on the Left the Dividend. Having finifhed this Divifion, you have found the integral Quote fought: And for the fractional Part, to the Remainder of the Divifion prefix all the Figures excluded from the Dividend; that is the true Remainder that would happen if the Divifion were done by the common Method: This Remainder, with the given Divifor, makes the Fraction. But obferve, that if there are any o's ftanding clear on the Right-hand of this Remainder, you may omit them all (in making your Fraction) if they do not exceed the Number of o's excluded from the Divifor; or as many of them, as are equal in Number to thofe in the Divifor; and omitting the fame Number of o's in the Divifor, of the remaining Figures on the Left make your Fraction.

Example 1. To divide 84700 by 4600, I divide 847 by 46, the Quote is 18, and 19 remains; but the true Remainder is 1900: Making this Fraction $\frac{1900}{4600}$, which is equivalent to this $\frac{19}{46}$.

Ex. 2. To divide 3640 by 800, I divide 364 by 8, the Quote is 45, and 4 remains; but the true Remainder is 420: Making this Fraction $\frac{420}{800}$, equal to this $\frac{42}{80}$.

Examp. 3. To divide 68704 by 2400, I divide 687 by 24; the Quote is 28, and 15 remains; but the true Remainder in the Queftion is, 1504, and the Fraction is $\frac{1504}{2400}$.

Examp. 4. To divide 367854 by 800, I divide 3678 by 8; the Quote is 459, and the Remainder 6; but the true Remainder is 654, and the Fraction $\frac{654}{800}$.

The *Reafon* of this Rule is contained in *Lem.* 4. Thus, if an equal Number of o's are excluded from Divifor and Dividend, the remaining Figures on the left exprefs like aliquot Parts of them, *viz,* a tenth Part if one o; a hundredth, if two o's were excluded, and fo on. But like aliquot Parts of two Numbers make the fame Quote as their Wholes, (by *Lem.* 4.) *Ex.* 46 and 847 are the hundredth Parts of 4600 and 84700, and fo have the fame Quote 18, with a Remainder 19, which is the hundredth Part of the Remainder in dividing 84700 by 4600; and therefore the two o's cut off from the Dividend are to be prefix'd to it, to give it the true Value. *Again,* if the Figures excluded the Dividend are not all o's, yet if we fuppofe them fo, the Quote is right: And that it cannot be increafed by the Value of the Figures cut off, whatever they are, is plain; be-

K

cauſe being prefix'd to the Remainder found by the Rule, they muſt make a leſs Number than the given Diviſor; ſince that Remainder is leſs than the Diviſor without its o's, which are as many in number as theſe Figures prefix'd to the Remainder, which therefore can never make up the Defect. So in *Ex.* 4. when 3678 is divided by 8, the Remainder is 6, which being leſs than 8, no Figures prefix'd to it can make a Number equal to 8, with as many o's prefix'd. But theſe Figures being a part of the Dividend, belong to the Remainder; which, inſtead of 600, as it would have been had the Figures cut off been o's, is 654.

Tho' this is the proper Demonſtration of this Rule, yet you may be ſufficiently ſatisfied of the Juſtneſs of it, by conſidering any Example wrought at length; wherein it will eaſily appear, that by excluding the o's in the Diviſor, and as many Figures in the Dividend, we only ſave the Trouble of writing many ſuperfluous Figures, and yet bring out the ſame Quote. As in theſe *Examples.*

$$4600)84700(18$$
$$4600$$
$$\overline{3800}$$
$$36800$$
$$\overline{1900}$$

$$800)367854(459$$
$$3200$$
$$\overline{4785}$$
$$4000$$
$$\overline{7854}$$
$$7200$$
$$\overline{654}$$

As to that part of the Rule for contracting the Fraction, you'll find the Reaſon of it explain'd in Book II.

CASE 3. If the Diviſor is the Product of two or more Digits, and that you can eaſily diſcover theſe Digits; then divide firſt by any one of theſe, and the Quote by any other, and ſo on: the laſt Quote is that ſought. And for the Fraction, multiply the Diviſor by the laſt Quote, and take the Product from the Dividend, you have the Remainder, which would happen by dividing after the common way. Or find it thus; Multiply the laſt Remainder (of the Work) by the preceding Diviſor (or the laſt but one) and to the Product add the preceding Remainder; this Sum multiply by the next preceding Diviſor, and to the Product add the next preceding Remainder, and ſo on, till you have gone thro' all the Diviſors and Remainders to the firſt. But when there are no Remainders in any of the particular Diviſions, the Dividend is a Multiple of the Diviſor.

Ex. 1. To divide 9048 by 24, I divide by 4 and 6, becauſe $4 \times 6 = 24$. Thus, $9048 \div 4 = 2262$, then $2262 \div 6 = 377$.

Ex. 2. To divide 754683 by 42, I divide by 6 and 7, becauſe $6 \times 7 = 42$, as in the Margin; wherein the firſt Quote is 125780, and 3 remains, which I

6 | 754683
7 | 125780³
  | 17968²⁷⁄₄₂

have ſet over a Line after the Quote; then the ſecond Quote is 17968, and 4 remains, which multiplied into the preceding Diviſor 6, produces 24, to which the firſt Remainder 3 being added, makes 27; ſo that the true Remainder is 27, and the fractional part of the Quote ²⁷⁄₄₂.

Ex. 3. To divide 18472 by 32, I divide by 4 and 8, becauſe $4 \times 8 = 32$; the firſt Quote is 4618, and nothing remains; the ſecond Quote is 577, and 2

4 | 18472
8 | 4618
  | 57⁻⁸⁄₃₂

remains, which multiplied into 4, produces 8, the true Remainder. But in this Caſe, where there is no Remainder in the firſt Step, the Fraction may be made of the Remainder, and Diviſor of the ſecond Step. So here it may be ²⁄₈.

Ex. 4. To divide 48767 by 15, I divide by 3 and 5, becauſe $3 \times 5 = 15$: the firſt Quote is 16255, and 2 remains; the ſecond Quote is 3251, and no-

3 | 48767
5 | 16255²
  | 3251²⁄₁₅

thing remains; wherefore there is no Product to be added to the firſt Remainder: and ſo that is the true Remainder, and the Fraction is ²⁄₁₅.

Ex.

Ex. 5. To divide 3428689 by 126, I divide by 3, 6, 7, becaufe 3×6×7=126: the firft Quote is 1142896, and the Remainder 1; the fecond Quote

3 | 3428689
6 | 1142896 $\frac{1}{}$
7 | 190482 $\frac{4}{}$
    27211$\frac{103}{126}$

is 190482, and the Remainder 4; the laft Quote is 27211, and the Remainder 5, which multiplied into the preceding Divifor 6, produces 30; to which the preceding Remainder 4 added, makes 34; which multiplied by the preceding and firft Divifor 3, produces 102; and the Remainder 1 added, makes 103 the true Remainder, and the Fraction is $\frac{103}{126}$.

The *Reafon* of this Rule is explained in *Lem* 3. and as to the Contraction of the Fraction in *Ex.* 3. you'll learn the reafon of it in Book II.

This Practice is of very good ufe, efpecially where the Divifor is the Product of two Digits; becaufe when it is fo, they are eafily difcovered: and the ufe of it you'll find more remarkably in the next Chap. §. 5. Obferve alfo, that if the Factors of the Divifor are 11 or 12, it's eafy to divide by them as by a Digit. Thus to divide by 144, chufe 12, 12, becaufe 12 × 12 = 144. For 33 take 3, 11. For 84, take 7, 12.

C A S E 4. One who is tolerably acquainted with the Practice of Divifion, according to the preceding general Rule, may contract the Work, by omiting to write down the Product of every Figure of the Quote by the Divifor; doing it in mind, and gradually as the Product is made, fubtracting it from the correfponding Figures of the Dividual; fetting the Remainder either above or below the Dividend, in the manner of the following Examples: For it's no matter whether the Figures of any Remainder or Dividual ftand all in one Line, if they are duly fituated with refpect to one another, from Left to Right-hand. Alfo, inftead of fetting the Quote on the Right-hand of the Dividend, it may ftand as conveniently under or over the Dividend.

<div align="center">

Ex. 1. 72849 ÷ 46 = 1583, and 31 remains.

</div>

That you may perceive the manner of working without Confufion, I fhall reprefent it as it appears feparately at every Step.

26	3	31	313
46)72849	268	2686	26861
1	46)72849	46)72849	46)72849
	15	158	1583 $\frac{31}{46}$ Quote.

The firft Dividual is 72, the Quote 1, and Remainder 26. The fecond Dividual is 268, the Quote 5, and Remainder 38; which is found gradually; Thus, 5 × 6 = 30; then 0 (the firft Figure of the Product) from 8 (the firft Figure of the Dividual) remains 8. Again 5 × 4 = 20, and 3 (the Number of 10's carried from the laft Product) is 23: then 23 from 26 (of the Dividual) remains 3; whence the next Dividual is 384; the Quote 8, and Remainder 16; Found thus, 8×6 = 48, then 8 (of the Product) from 4 (of the Dividual) cannot be taken, but from 14, and 6 remains: Again 8 × 4 = 32, and 4 (from the laft Product) is 36; then 36 from 37 (inftead of 38 of the Dividual, becaufe 1 was taken from the 8 in the laft Step to make 14) leaves 1. Or it comes to the fame thing, if to the Product we add 1, for the 10 that was borrowed in the laft ftep; fo the Product 36 and 1 (borrowed) is 37; which taken from 38, there remains 1. The next Dividual is 169, the Quote 3, and Remainder 31.

By this Example you may underftand how to do, or examine others. See the following.

	33			3			23
	4304	Or it may		304		•	2527
Ex 2.	68)24786	ftand thus.	68)24786				445545
Quote.	364 $\frac{54}{68}$		364 $\frac{54}{68}$		*Ex.* 3.	467)1247893	
						Quote.	6954$\frac{175}{467}$

<div align="center">K 2</div>

Obferve,

Obſerve, When in any Step there is 10 borrowed in the Subtraction, then the Remainder will be always greater than the Subtrahend, (becauſe if any Digit is taken from the Sum of 10 and a leſſer Digit, the Remainder muſt be greater than that leſſer Digit.) And when you come to the next Step, look back upon the laſt Remainder, its being greater than the Subtractor over which it ſtands, is a certain ſign that 10 was borrowed in the laſt Step, and conſequently that 1 for that 10 is to be taken from the Subtractor, or added to the Product in this Step: And this Obſervation is very uſeful, becauſe we are apteſt to forget this 1. So in *Ex.* 3. the laſt Dividual is 2243, the Quote 4, and Remainder 375: Thus found, $4 \times 7 = 28$; then 8 from 13 leaves 5. Again, $4 \times 6 = 24$, and 2 (carried from the laſt Product) is 26, and 1 (borrowed, becauſe 5 is greater than 3 over which it ſtands) makes 27; then 7 from 14 leaves 7. Again $4 \times 4 = 16$, and 2 (carried from the laſt Product) and 1 (borrowed) makes 19, which taken from 22, leaves 3.

In the common way of practiſing this Method, they daſh a Line thro' every Figure of the Dividual, gradually as the Figures of the Remainder are ſet over them, in order to prevent Confuſion; becauſe the next Dividual appears the more diſtinctly from the Figures of the preceding Dividuals that are thus cancelled. As in this *Example,* repreſented in all its Steps ſeparately.

9	2	21
24)5726	90	924
2	24)5726	24)5726
	23	238 14/24

Obſerve, As it is to the ſame Purpoſe whether we write the Remainders over or under the Dividend, ſo the Method of placing them has a little Confuſion in it, which is helped by daſhing the Figures. But I think it a more convenient way to ſet the Remainders under the Dividuals, and ſo as the Figures of the ſame Remainder be in a Line; leaving the next Figure of the Dividend, which makes up the Dividual, where it ſtands, and ſetting the Quote over the Dividend, as in theſe *Examples.*

```
  238  Quote.
24)5726  Dividend.
  9  ⎫
 20  ⎬ Remainders.
 14  ⎭
```

I ſhall recommend this Method of Diviſion only to ſuch as, by Practice, have acquired a Habit of cloſe and careful Attention; otherwiſe 'tis too difficult. But there are ſome particular Circumſtances wherein 'tis very eaſy and convenient, as in the two following *Caſes.*

```
  364  Quote.
68)24786  Dividend.
 43  ⎫
 30  ⎬ Remainders.
 34  ⎭
```

C A S E 5. If the Diviſor has the Figure 9 in all its Places, as 9, 99, 999, &c. then the Quote is either 1, when the Dividual is equal to the Diviſor; or if the Dividual has one place more than the Diviſor, the Quote is either the firſt Figure on the Left of the Dividual, or the next greater Figure. Thus, if the firſt Figure added to the remaining Figures (taken as one Number) makes a Sum leſs than the Diviſor, that firſt Figure is the Quote, and that Sum is the Remainder, after the Product of the Diviſor and Quote is taken out of the Dividual. But if that Sum is equal to, or greater than the Diviſor, the Difference is the Remainder, and the Quote is the Figure next greater than the firſt Figure of the Dividual.

Thus the Quote and Remainder being eaſily found, (without the Product) we may chuſe the Method of the preceding *Caſe* for placing them; and we may alſo make it ſhorter, by not writing down ſuch Figures of the Remainder which are the ſame with their Correſpondents in the Dividual, but leaving them where they ſtand.

Ex.

Ex. 1.   35
          72428
99)468793
Quote. 4735 $\frac{28}{99}$

Ex. 2. o
          92 4
999)67934582
Quote. 68002 $\frac{584}{999}$

Here the firft Dividual is 468, the Quote 4, and the Remainder 72 (=68+4.)   The next Dividual is 727, the Quote 7, and Remainder 34 (=27+7.)   The next Dividual is 349, the Quote 3, and Remainder 52 (=49+3.) The next Dividual is 523, the Quote 5, and Remainder 28.

Here the firft Dividual is 6793, the Quote 6, and Remainder 799 (=793+6.)   The next Dividual is 7994, the Quote is 8 (=7+1,) and the Remainder 2, (becaufe 994+7=999 +2.)   The next Dividual 25, and the next again 258, being each lefs than the Divifor, the Quote Figures are o; the laft Dividual is 2582, the Quote 2, and Remainder 584 (=582+2.)

By thefe Examples you may eafily do or examine any other; fee the following.

D E M O N S T. The thing to be demonftrated in this *Rule,* is the way of finding the Quote and Remainder when the Dividual has one place more than the Divifor, which, upon a little Attention, will be very obvious.   For 99 wants 1 of 100, (and any Number expreffed thus by 9's, wants 1 of a Number expreffed by 1, and as many o's.)   And however oft 100 is contained in any Number of the fame Number of places, (which is always as oft as the firft Figure on the Left expreffes, the remaining Figures on the Right being the Remainder; fo 462 ÷ 100 = 4, and 62 remains) fo oft at leaft muft 99 be contained in it: and the Remainder will be the Sum of the Remainder when divided by 100, and of the Quote or firft Figure; which is the Remainder when 99 is taken out of any Number of Hundreds.   Thus, 468 ÷ 99 = 4, and 72 remains, *viz.* 68+4; for in 400 ÷ 99, the Remainder is 4.   And therefore, fince 4+68 is lefs than 99, it muft be the true Remainder in dividing 468 by 99.   *Again,* 697 ÷ 99 = 7 and 4 remains; for 99 is contained in 600 fix times, and 6 remains; then 6+97=99+4, in which 99 is contained once, and 4 remains: Wherefore 99 is contained in 697, 7 times, and 4 remains.   The fame Reafoning holds in all Cafes.

Examp. 3.
          73
         7622
999)746985
Quote. 747 $\frac{212}{999}$

Examp. 4.
         3 9
99)39976
Quote. 403 $\frac{79}{99}$

Examp. 5.
          92 4
999)67934582
Quote. 68002 $\frac{584}{999}$

Examp. 6.
          71
99)99467
Quote. 1004 $\frac{71}{99}$

Examp. 7.
         9 61
99)6039457
Quote. 61004 $\frac{61}{99}$

Examp. 8.
         9
99)40596
Quote. 410 $\frac{6}{99}$

Examp. 9.
          809
99)67419
Quote. 681

The Ufe and Conveniency of this *Cafe,* you'll find chiefly in *Chap.* 4. *Book* 5. where fuch Divifors frequently occur.

C A S E 6. If the Divifor has any other Figure than 9 in all its Places, as 44, 666, *&c.* doing the Work in the manner of *Cafe* 4, will be eafier than if the Figures were all different: But it will be ftill eafier with a little more Work.   Thus, divide by 11 or 111, *&c.* with as many Places as the Divifor has; and here the Quote is eafily feen, for it is either the firft Figure on the Left of the Dividual, or the next leffer Number; or o, if the Dividual has the fame Number of Places as the Divifor.   But if the Dividual has one Place more than the Divifor, the Quote is 9.   Then the Product of the Divifor and Quote having the fame Quote Figure in all its Places, the Remainder is eafily found without writing down the Product, and fo may be difpofed in the manner of *Cafe* 4.   Then, laftly, divide this Quote by that Digit of which the Divifor confifts, by *Cafe* 1.

Ex.

Examp. 1.   44)562387.   *Examp.* 2.   777)4672869.
      Thus.   Thus,

$$11261$$
$$11)562387$$
$$4)\ 51126\,\tfrac{1}{\ }$$
$$12781\tfrac{23}{44}$$

$$1\ \ 10$$
$$230872$$
$$111)4672869$$
$$4)\ \ 42097\,\tfrac{102}{\ }$$
$$10524\tfrac{211}{777}$$

A General Method *to make* DIVISION *certain and eafy.*

The Work of *Divifion* may be made more certain and eafy with the writing a few more Figures; by making a Table of the Products of the Divifor by all the Digits; as was done in §. 2. of the preceding *Chap.* for Multiplication. To be ufed thus; Seek the Dividual, or the next leffer Number, in the Table againft it is the Quote Figure, and that Number itfelf is the Product of the Divifor and Quote: By this means the Work may be done as faft as Figures can be written. Nor need the Products be copied out of the Table, but taken where they ftand. The Remainders may be found and written under the Dividual: And here alfo we may chufe to write the Dividuals and Remainders under the Dividend, and without bringing down the Figures of it to the Remainders, taking them where they ftand, as has been fhewn in *Cafe* 4.

Example.

1	483	)35268749(72560 $\frac{69}{483}$
2	986	3401
3	1469	1258
4	1952	986
5	2435	2727
6	2918	2435
7	3401	2924
8	3884	2918
9	4367	69

Here the firft Dividual is 3526; the neareft Number to it in the Table is 3401, againft 7, which is therefore the Quote, and 3401 is the Product. The next Dividual is 1258, whofe next Number in the Table is 986, &c.

But the Work may be made without writing the Products out of the Table; and it will ftand thus.

1	483	)35268749(72560 $\frac{69}{483}$
2	986	125
3	1469	272
4	1952	292
5	2435	6
6	2918	
7	3401	
8	3884	
9	4367	

SCHOLIUM. As to this Method of Practice, the Readinefs with which the Quote and Product is found, may be reckoned a fufficient Balance for the Time and Trouble of making the Table, in all Cafes where the Divifor exceeds two Places, and the Dividend is fo large a Number that the Quote will confift of feveral Figures; and efpecially too, if we fubtract the Products as they ftand in the Table, without writing them under the Dividual: And this part will ftill be more eafy, if we write the Table upon a feparate bit of Paper, fo as we can place each Product under the Dividual.

And, laftly, in fuch Queftions and Calculations where the fame Number muft be frequently ufed as a Divifor, this Method of a Table, efpecially a moveable one, is moft convenient, as you'll find afterwards.

NEPER's RODS may alfo be ufed for making the Table of the Divifor.

Of the P R O O F S of D I V I S I O N.

1. We have already explained one *Proof* of *Divifion* by *Multiplication*, which is this; *viz.* The Product of the Divifor and Integral Quote added to the Remainder, is equal to the Dividend.

2. But it may be alfo proved by *Divifion.* For the *Dividend* being divided by the Integral Quote, the *Quote* of this *Divifion* will be equal to the former *Divifor*, with the fame *Remainder.* Thus, 3 is contained 4 times in 14, and 2 remains: But 4 times 3 = 3 times 4; therefore 4 muft be contained 3 times in 14, with the fame Remainder 2 ; as it actually is. The fame Reafon is good in all Cafes.

3. Laftly, *Divifion* may be proved by cafting out the 9's. Thus; Subtract the *Remainder* out of the *Dividend*, what remains here ought to be the Product of the *Divifor* and *Quote* ; which you may prove by cafting out the 9's, as was done in *Multiplication*.

C H A P. VII.

Of Applicate Numbers.

Explaining the preceding Fundamental Operations, as they are Applicable to Queftions *about* Particular Things, *with their Circumftances in Human Affairs.*

§. 1. *Of* Applicate Numbers ; *and their* Diftinction *of* Simple *and* Mix'd.

With TABLES *of the Variety of* Coins, Weights, *and* Measures *of* G R E A T B R I T A I N.

WHY Numbers are called *Applicate*, we have learned already. But now we are to confider, That for the Ufe of Society, it was neceffary that certain greater Quantities fhould be fubdivided into other leffer ones; and thefe again into others leffer; each having its diftinct and proper Denomination ; but all confidered as fubordinate Species of the greater; in order to the giving and receiving more or lefs of any Goods or Commodity, as occafion fhould require. As for Example, One Pound (of Money) is divided into 20 Shillings, one Shilling into 12 Pence, and one Penny into 4 Farthings. Thefe feveral leffer Quantities, as they have diftinct Denominations, are as really *Integers* of their own kind as the greater, of which they are a Part; and a Number of each Species confidered by itfelf, is called a *fimple Number*; as 48 Pounds, or 56 Shillings, &c. But when we take together a Number of feveral Species, taking ftill lefs of each inferiour Species than what makes an *Unit* of the higher, and confidering it as a Part of that *Unit*, this makes a *mix'd Number.* For Example; 48 *l.* 14 *s.* 9 *d.* 2 *f.* is one mix'd Number. Again confider, that as the Numbers of the inferiour Species that make a mix'd Number are lefs than an Unit of the fuperiour, and have always a known and

cer-

certain Relation to them, (as 1 Shilling is a 20ᵗʰ Part of a Pound;) so they are in effect *Fractions*, and being summed up with a regard to that relation (as we shall learn) they are truly considered as *Fractions*, (or Numbers related to one another) in the Operation. But the relative Denomination (which in every Fraction is a Number) being suppress'd and understood, [tho' considered in the Operation] and each Species distinguished by proper Denominations, which of themselves express no such Relation; they are all considered as *Whole Numbers*: as indeed each Species is in the most strict and proper sense, considered as an Integer with respect to the lower. So that each of them, except the highest and lowest, is considered both as a *Whole Number* and a *Fraction*.

This Account of the Nature of *mix'd Numbers* might perhaps be sufficient; yet there are several Reflections, useful to such as would have complete Notions of Things, concerning the Nature of the various Kinds of *mix'd Numbers*, that may be very proper in this Place: for tho' it may be thought a Digression from the business of *Arithmetical Operations*, yet it can never be impertinent to make useful Reflections upon the Subject of these Operations.

The Nature and Design of Society has introduced among Men a Necessity of exchanging such things as are the Product of their different Applications and Labour; for every Country does not produce, nor every Man apply himself to every thing: Now, whatever way this Exchange is made; whether things are valued by their real Use, or by Fancy, there is always some Equality supposed, or made by agreement of Parties, betwixt certain Quantities of one thing and another. And that Commerce might be regular and certain, it was necessary to constitute some fixed and standard Quantities under certain and constant Names; otherwise Men could never be able to treat about these Exchanges unless they were together, and the Subjects were immediately before them; and even then not without great Inconveniency. Again, because Men need less and more on different occasions, it was necessary there should be various Quantitys of every Kind, which differing only as *less* or *more*, it was convenient that each (or several) of the greater should contain a precise Number of the lesser as distinct and certain Parts of them, whereby subordinate Quantitys coming under one general Name, constitute one kind of *mix'd Number*; the several Parts or Denominations of which we call the several Species of that Kind.

The more common Subjects of these *mix'd Numbers* are the external sensible Objects which we see and feel, and which in general we call Bodies.

Now the Quantity of Bodies can only be considered in two respects, either as to their *Bulk*, i. e. the external Measure of *Length*, *Breadth*, and *Thickness*; or their *Weight*: And to compare different Bodies in these respects, there must be certain common standard *Measures* and *Weights* to which all others are compared. In some Bodies only one Dimension, which we call *Length*, is considered; because the other two are either inconsiderable in themselves; or rather, because in a comparison of more and less of these things, the other Dimensions are equal, (or supposed to be so;) and the Dimension chosen is that which admits of most Degrees: so that the more and less are here according to the *Length*. Hence proceed what we call the *Measures of Length*. See the following Tables of *mix'd Numbers*. Again, in some, Length and Breadth are both considered; and from this proceed the *Superficial* or *Square* Measures. Others are measured in all their three Dimensions; hence the *Solid* and *Cubical* Measures: under which may be comprehended what we call the *Measures* of *Capacity*, by which are measured the Quantity of *Liquids*, and of all such things as consisting of small Particles either altogether distinct, or cohering very loosely, cannot be measured singly, nor given out by Number; but are considered according to the *Bulk* they make, when being laid together they appear as one continuous Body; for Example, Corns, and Meal, or any solid Body reduced to Powder.

Again, other things are more conveniently measured by their *Weight*.

In the next place we must *observe*, That some things are exchanged by Number, the Individuals (which must all be of one Species of things) being really separated and distinct;

ftinct; and which having neither fuperiour nor inferiour Species, are not valued by Weight or Meafure; or one of them being fo valued, the reft are fuppofed to be equal; or the things are fuch, as cannot be weighed or meafured. In this manner are Cattle, and innumerable other things bought and fold by *Tale*, (in the common Phrafe;) and for thefe there is no diftinct Order of *mix'd Numbers*. But there is alfo a kind of *mix'd Numbers* conftituted for fome things that are exchanged by Tale; the Species of which are called *Grofs* and *Dozen*, &c. Now here the Species are not any real continued Quantitys, but certain Numbers diftinguifhed by particular Denominations, which therefore require no Standards, but to have a true Idea of the Number they are affixed to. Whereas in other things, the feveral Denominations give us immediately the Idea of fome continued Quantity; and we apply Number to them only by an arbitrary Subdivifion into Parts: So that we may conclude, that *mix'd Numbers* arife more generally and properly from the imagined Parts of continued Quantitys, either *Solids*, *Superficies*, or *Lines*.

As to Co i n s, *obferve*, That they are properly meafured by Weight: But the Weight being afcertained by publick legal Marks, their Species have not the proper Denomination of Weight; and therefore we don't ordinarily talk of them as things *weighed*; yet when there is any fufpicion of falfe Weights, they are compared to fome ftandard Weights.

As Numbers are not only applied to Bodies, and their imagined Parts, but alfo to every thing that is capable of *more* and *lefs*; as to the conceivable Parts of Time; fo we have alfo from this laft a particular kind of *mix'd Number*.

Again *obferve*, That for different things we have different kinds of Weights and Meafures; fo we have *Troy Weight* and *Averdupoife Weight*, &c. We have different Meafures for *Corn*, *Beer*, *Wine*, &c. Wherein thefe different *Weights* and *Meafures* coincide and agree, or what the Relation betwixt them is, and by what means their Standards were firft fettled, is not fo ftrictly the bufinefs of this Work to confider. The Statutes explain and determine thefe things; and perhaps Cuftom only is the Foundation of fome of them.

The laft thing I fhall obferve upon this Subject, is, That of the Denominations of *Coins*, *Weights*, and *Meafures*, fome are merely imaginary, *i. e.* are not Names of any one real diftinct Quantity, but of fome poffible Quantity fuppofed equal to a certain Number, or a certain Part of fome real ftandard Quantity. So for Example, a *Pound of Money* is an imaginary Quantity equal to 20 Shillings. A *Laft* is an imaginary Quantity equal to 12 Barrels. And again, you muft *obferve*, that there are many more Denominations known, and ufed upon different Occafions in treating or fpeaking of thefe things, than are convenient or ordinarily ufed in keeping Accounts. In the following *Tables* I fhall give you a full account of all the Kinds and Denominations that are commonly known; and diftinguifh thofe that are ufed in keeping Accounts.

T A B L E S *of the moft common* **Co i n s, We i g h t s,** *and* **M e a s u r é s,** [*Real and Imaginary*] *of* **G R E A T   B R I T A I N.**

Englifh Money.

4 Farthings		1 Penny.
4 Pence		1 Groat.
6 Pence		1 Tefter.
12 Pence		1 Shilling.
5 Shillings	=	1 Crown.
6 Shillings + 8 Pence		1 Noble.
10 Shillings		1 Angel.
13 Shillings + 4 Pence		1 Mark.
20 Shillings		1 Pound.

The *Real Coins* now Current and commonly known, are thefe:

1. Of Copper-Money; a Farthing, and a Halfpenny.

2. Of Silver-Money; a Penny, Twopence, Four-pence, Six-pence, a Shilling, Half a Crown, a Crown.

3. Of Gold-Money; Half a Guinea = 10 Shillings + 6 Pence; a Guinea = 21 Shillings, or 1 Pound + 1 Shilling.

There are many other Gold-Coins, and fome Silver, but not very common.

Ac-

Accounts are kept in Pounds, Shillings, Pence, and Farthings; which are marked by thefe Characters, $l : s : d : q$ or f. Whofe Relations I mark over them, thus;

$$\overset{20}{l} : \overset{12}{s} : \overset{4}{d} : f. \quad i.\ e.\ 4f = 1\ d.\quad 12\ d = 1\ s.\quad 20\ s = 1\ l.$$

Obferve, In *Scotland* they ufe the fame Denominations, except Farthings, and 1 Pound *Englifh* = 12 Pound *Scotch.* But they begin now to ufe *Englifh* Money in their Accounts.

ENGLISH WEIGHTS.

Troy Weight.

24	Grains	1 Penny-weight.
20	Penny-weight =	1 Ounce.
12	Ounces	1 Pound.

Accounts are kept in the fame Denominations, marked thus:

$$\overset{12}{\mathit{lb}} : \overset{20}{oz} : \overset{24}{dw} : gr.$$

Averdupoife Weight.

4	Quarters of a Dram	1 Dram.
16	Drams	1 Ounce.
16	Ounces	1 Pound.
14	Pound =	1 Stone.
28	Pound	1 Quart. of a Hund.
4	Quarters of a Hundred	1 Hundred Weight.
20	Hundred	1 Tun.

Apothecary's Weight.

20	Grains	1 Scruple.
3	Scruples	1 Dram.
8	Drams =	1 Ounce.
12	Ounces	1 Pound.

Marked thus.

$$\overset{12}{\mathit{lb}} : \overset{8}{\mathfrak{z}} : \overset{3}{\mathfrak{z}} : \overset{20}{\mathfrak{D}} : gr.$$

In keeping Accounts, this Weight is fub-divided into two Kinds, called *Averdupoife Weight, the Greater,* and *the Leffer.*

The Greater comprehends thefe Denominations, Tun. Hundred Weight. Quarter.

Pound. Marked thus; $\overset{20}{T.}$ $\overset{4}{C.}$ $\overset{28}{Qr.}$ $\mathit{lb}.$ But the laft 3, *viz.* C. Qr. *lb.* are fufficient.

The leffer comprehends thefe; Stone. Pound. Ounce. Dram. Quarter. Marked thus: $\overset{14}{St.}$ $\overset{16}{lb.}$ $\overset{16}{oz.}$ $\overset{4}{dr.}$ qr.

Obferve, In *Scotland* the Stone is commonly reckoned 16 Pounds.

The *Original* of all Weights in *England* was a *Corn* of Wheat taken out of the Middle of the Ear, and well dried; of which 32 made one Penny-weight; inftead of which, they made afterwards another Divifion of the Penny-weight into 24 Grains. Mr. *Ward* (in his *Young Mathematician's Guide,*) cites a Statute of *Edward* III. by which there ought to be no Weight ufed but *Troy.* But Cuftom, fays he, afterwards prevailed in giving larger Weight to coarfe and droffy Commodities, and thereby introduced the Weight called *Averdupoife,* And as to the Proportion betwixt *Troy* and *Averdupoife* Weight, he fays, That by a very nice Experiment he found that 1 Pound *Averdupoife* is equal to 14 Ounces, 11 Penny-weight, 15 and ½ Grains *Troy.* So that neither the Ounce nor Pound are the fame.

By *Troy* Weight are weighed *Jewels, Gold, Silver,* and *Bread.*

By *Averdupoife* Weight are weighed all *Grocery* Wares.

The Apothecary's Pound is *Troy* Weight: but inftead of fubdividing the Ounce into *dw.* they divide it into Drams and Scruples.

Sheeps Wool Weight has thefe Denominations; 7 Pounds = 1 Clove: 2 Cloves = 1 Stone: 2 Stones = 1 Tod: 6½ Tods = 1 Wey: 2 Weys = 1 Sack: 12 Sacks = 1 Laft.

LIQUID

LIQUID MEASURE.

Wine Meafure.

2 Pints	1 Quart.
4 Quarts	1 Gallon.
42 Gallons	1 Tierce.
1 ½ Tierce	1 Hogfhead.
1 ½ Hogfhead	= 1 Punchion.
1 ½ Punchion or ⎱ 2 Hogfheads ⎰	1 Butt or Pipe.
2 Butts or Pipes	1 Tun.

Which may for Accounts be reduced to thefe; Tun. Hogfhead. Gallon. Quart. Pint. Thus Marked:

$$4 \quad 63 \quad 4 \quad 2$$
$$T : hd : gal : qt : pt.$$
$$63 \quad 8$$

Or it may be fufficient to ufe hd : gal : pt.

Obferve alfo, That all Spirits, Mead, Perry, Cyder, Vinegar, Oil, and Honey are meafured as Wine. Again, 18 Gallons make 1 Runlet; and 31 ½ Gallons make a Wine or Vinegar Barrel.

Ale and Beer Meafure.

2 Pints	1 Quart.	⎫ A Firkin
4 Quarts	1 Gallon.	⎪ of Soap
8 Gallons Ale ⎱	= 1 Firkin	⎪ andHer-
9 Gallons Beer ⎰		⎬ rings are
2 Firkins	1 Kilderkin.	⎪ the fame
2 Kilderkins	1 Barrel.	⎪ with that
1 ½ Barrels	1 Hogfhead.	⎭ of Ale.

Mr. *Ward* fays, This Diftinction of the Ale and Beer Meafures are now ufed only in *London*; but in all other Places of *England* it is by a Statute of Excife made in the Year 1689, without diftinction 8 ½ Gallons to 1 Firkin. And this Meafure may be reduced to thefe Denominations, *viz.*

$$48 \quad 4 \quad 2$$
$$hd : gall : qt : pt. \text{ for Ale.}$$
$$54 \quad 4 \quad 2$$
$$hd : gall : qt : pt. \text{ for Beer.}$$

Or, according to the laft Account for both,
$$51 \quad 4 \quad 2$$
$$hd : gall : qt : pt.$$

There is alfo another way of keeping Accounts, efpecially in the Affairs of the *Revenue*, as the *Excife*; where they make the loweft Denomination a Cubical Inch, (*i. e.* a Meafure 1 Inch long, 1 Inch broad, and 1 Inch deep:) And then of Wine Meafure, 231 Cubick Inches make 1 Gallon. In Ale and Beer Meafure, 282 Cubick Inches make 1 Gallon; and you may chufe as many of the fuperiour Denominations in your Accounts as you pleafe.

As to the Original of Liquid Meafure, it is from *Troy* Weight. Thus; 8 ℔ *Troy* Weight of Wheat gathered out of the Middle of the Ear, and well dried, is, by the old Statutes of *Henry* III. *&c.* ordained to be a Gallon of Wine Meafure; neither were any other Meafures allowed, tho' Time and Cuftom has introduced others. Mr. *Ward* mentions an Experiment he was witnefs to at *Guildhall*, before the Lord-Mayor of *London* and others; whereby it was found that the old Standard Wine Gallon contained exactly 224 Cubical Inches; tho', fays he, for feveral Reafons, the fuppofed Content of 231 Inches was continued.

Obferve, In *Scotland*, the common Denominations of Liquid Meafure are thefe: Hogfhead. Gallon. Pint. Mutchkin. Gill. and 4 Gill = 1 Mutchkin; 4 Mutchkins = 1 Pint; 8 Pints = 1 Gallon; and 16 Gallons = 1 Hogfhead. They alfo call 2 Mutchkins, 1 Chopin; and 2 Pints 1 Quart. The *Englifh* Pint is a very little larger than a *Scotch* Mutchkin. But the Excife in *Scotland*, fince the Union of the two Nations, is calculated upon *Englifh* Meafure.

DRY MEASURE, *called* *also* CORN MEASURE.

2 Pints	1 Quart.
2 Quarts	1 Pottle.
2 Pottles	1 Gallon.
2 Gallons	1 Peck.
4 Pecks	1 Bufhel, Corn.
5 Pecks =	1 Bufhel, Water.
4 Bufhels	1 Coomb.
2 Coombs	1 Quarter.
4 Quarters	1 Chalder.
5 Quarters	1 Tun or Wey.
2 Weys	1 Laft.

For Accounts, thefe Denominations are fufficient;

$$4 \quad 8 \quad 4 \quad 16$$
Ch : qr : bufh : pk : pt.

I have taken this Table as it is in *Wingate* and others; but *Ward* fays, 10 Quarters = 1 Wey, and 12 Weys = 1 Laft of Corn.

As in Liquid Meafure, fo in dry, the loweft Denomination ufed in the Calculations of the Revenue is a Cubic Inch, whereof $268 + \frac{4}{5}$ make 1 Gallon: For the *Winchefter* Bufhel with a plain round Bottom and equally wide from Top to Bottom, is $18 \frac{1}{2}$ Inches wide, and 8 Inches deep; whence follows by Calculation, that $268 \frac{4}{5}$ Cubical Inches make 1 Gallon.

The common Denominations of Corn Meafure in *Scotland*, are Chaldron. Boll. Bufhel.

$$\overset{16}{\text{Boll.}} \quad \overset{4}{\text{Bufhel.}}$$

$$\overset{4}{\text{Peck.}} \quad \overset{4}{\text{Quarter.}}$$ But they are different Meafures from the *Englifh* of the fame Name.

MEASURES *of* LENGTH.

1ft.

12 Inches	1 Foot.
3 Feet	1 Yard.
45 Inches	1 Eln.
2 Yards =	1 Fathom.
$5\frac{1}{2}$ Yards	1 Pole or Perch.
40 Poles	1 Furlong.
8 Furlongs	1 Mile.

For Accounts ufe thefe Denominations;

$$8 \quad 220 \quad 3 \quad 12$$
Mile : furl : yd : feet : Inch.

2d.

4 Nails = 1 Quarter
4 Quar = 1 Yard.

Marked

$$4 \quad 4$$
yd : qr : na.

The Original of Long Meafures is from a Corn of Barley, whereof 3 taken out of the Middle of the Ear, and well dried, make 1 Inch; and therefore 1 Barley-Corn is the leaft Meafure, but not ufed in Accounts.

TIME.

60 Seconds	1 Minute.
60 Minutes	1 Hour.
24 Hours =	1 Day.
365 Da. + 5 ho. + 48 min. + 57 fec. }	1 Year.

Becaufe 1 Day is not an *aliquot* Part of a Year, therefore all thefe Denominations cannot conveniently be mixed in Accounts. And we may chufe to make Days the greateft Denomination in Accounts; and then any Number of Days may be again reduced to Years, by dividing them by 365 Days, 5 Hours, 48 Minutes, 57 Seconds, as will be afterwards taught. In Aftronomical Calculations there is a neceffity to be thus exact: But for common Ufes we may neglect the 5 Hours, 48 Minutes, 57 Seconds, and make 365 Days = 1 Year. Or alfo cafting away 1 Day + 5 Hours, &c. we may call 364 Days = 1 Year: And make this Divifion of the Year, *viz.*

60 Seconds	1 Minute.
60 Minutes	1 Hour.
24 Hours	1 Day.
7 Days	= 1 Week.
4 Weeks, or } 28 Days }	1 Month.
13 Months	1 Year.

But to make 365 Days = 1 Year, and use no Denomination betwixt Year and Day, is the better way in Calculations of Intereft, where it occurs moft; as you'll find afterwards: yet Months and Weeks make a convenient Divifion of Time.

Of SUPERFICIAL or SQUARE MEASURE.

16 Square Quarters } of an Inch }	1 Square Inch.
144 Square Inches	1. Square Foot.
9 Square Feet	= 1 Square Yard.
30¼ Square Yards	1 Square Pole.
40 Square Poles	1 Rod of Land.
4 Rods	1 Acre.

Obferve, Square Meafure is that which is as long as broad; and therefore as 4 Quarters make 1 Inch in Length, fo a Surface 1 Inch long and 1 Inch broad is divifible into 16 Parts, each ¼ Inch long and broad; and fo of the reft. The Reafon of which will be underftood after you know what Multiplication is.

For fmall Surfaces the Denominations of inch. foot. yd. are enough. And for Land thefe of Acre. Rod. Pole. Alfo when we fay fo many fquare Feet or Yards, &c. it were the fame thing to fay fo many Feet or Yards long, and one broad. And thus a Rod is 40 Poles (or 220 Yards) long, and 1 Pole (or 5¼ Yards) broad; which is alfo 1210 Yards long, and 1 broad.

SCHOLIUM, *relating to the following* Applications.

Thofe who would be very nice and fcrupulous as to the Method and Order of bringing in the following *Applications,* would firft explain all the four fundamental Operations of *Addition, Subtraction, Multiplication,* and *Divifion,* as applied to fimple Numbers, before they fay a word of mixed Numbers; and they have this Reafon for it, *viz.* Becaufe the Addition of mixed Numbers is indeed not the fimple Effect of Addition, but of that and Divifion too: and therefore, according to the ftricteft Method, ought to be brought as a mixed Application of both. It is true, that according to the Order we have hitherto followed, (the general Rule of Divifion being already taught) we may propofe any Rule with a Divifion to be performed; for this cannot be called the propofing to one the doing of a thing which he has not yet learned: yet ftill it will have thus much of *Diforder* in it, that we fhall anticipate fomething that does more immediately and properly belong to the Application of *Divifion.* But that being fimple, and the Reafon of it very obvious, I have chofen rather to explain the Practice of *Simple* and *Mixed Numbers* one immediately after the other, in each of the four Operations. And to ferve thofe who incline to learn the Addition of Applicate Numbers before they learn Divifion of Abftract Numbers, I have alfo explained the Method neceffary to be ufed in that Cafe.

§. 2.

§. 2. ADDITION *of* APPLICATE NUMBERS.

CASE I. *To add* Simple Numbers *all of one Denomination.*

RULE. This is done in all refpects as *Abftract Numbers,* thefe being no other than *Numbers applicable to any kind of thing.* So that a particular Application can never alter the Rule and Reafon of Operation. And that they ought to be all of one Denomination, is alfo plain. But fee the following *Scholiums.*

SCHOLIUMS.

L. Sterling.	Years.
2468	347
7890	256
5678	789
2345	563
L. 18381	Sum Y.1955

1. That is called a *Proper Addition,* when Numbers are fo added together as that their Sum is a new Number diftinct from either of them; and that *Applicate Numbers* may be added in this manner together, the *Units* of each that are to be added muft be of the fame Value and Denomination, or applied to the fame Species of things, elfe the Sum can have no particular Denomination, and fo be of no ufe in Practice. So 4 Men and 3 Books make 7 things; but are not either 7 Men or 7 Books.

Again, Numbers of different Denominations are faid to be added *improperly* when each Number is diftinctly reprefented by itfelf with fome Mark or Word for Addition; as for Example, 3 *L* and 4 *s.* or 3 *l.* + 4 *s.* or more fimply 3 *l* : 4 *s.* and fuch Additions conftitute *mixed Numbers.* But ftill it is to be minded, that *applicate Numbers* cannot be added, even improperly, *i. e.* to make one *mixed Number,* unlefs they have certain Relations to one another, fo that a certain Number of one kind is equal to one of another. And for the fame Reafon when there are feveral *mixed Numbers* to be added, the Numbers of each Species muft be feparately and diftinctly added together, as in the following Cafe: Concerning which, carefully obferve the following Article.

2. The Sum of feveral *mixed Numbers* may be found and exprefled after two very different manners. (1.) We may add the Numbers of every Species by itfelf into one complete Sum; and exprefs thefe feveral Sums diftinctly and feparately. Or, (2.) Regarding the mutual Relations of the feveral Species, the total Sum may be found and exprefled in a more fimple manner; fo that there fhall be no Number in it of any inferiour Species but what is lefs than an *Unit* of the next above. [The Value of the Sums of each inferiour Species being exprefled in Numbers of the next higher Species, gradually to the higheft.]

Now the *Firft* is the only Method natural and proper to *Addition* ; for the Anfwer it finds is the pure Effect of *Addition;* and is indeed only fo many diftinct Queftions of fimple Numbers added, without any dependence or regard to one another. As in the annex'd *Example.*

l. : *s.* : *d.*
24 : 14 : 10
68 : 18 : 09
352 : 06 : 11
467 : 10 : 08
1ft Method 911 : 48 : 38
2d Method 913 : 11 : 02

But the Anfwer found by the fecond Method, (which fhall be taught immediately,) is the moft Simple and Ufeful in Bufinefs; becaufe it exprefles the Whole in one Species (and that the higheft) as near as poffible; and fo makes the Comparifon of different Quantities more fimple and eafy: yet the Operation is more complex; for it is the mixed Effect of *Addition* and *Divifion.* I have in this *Example* exprefled the Sum both ways. How the fecond is performed, you learn in the following *Rule.*

I

Laftly,

Laftly, Obferve, That tho' the Numbers to be added are all of one Denomination, and fo far belong to the firft *Cafe*; yet if they are of fuch kind of things as have fuperior Species, the Anfwer may, in fome Cafes, be found and expreffed two ways; either in their own Species by fimple Addition, as in *Cafe* 1. or regard may be had to the fuperior Species: And then, if their Sum taken in their own Species is greater than an Unit of the Superiour, the total Value of the given Numbers may be expreffed in the fuperiour Species, as far as it reaches. And fo it will be either a fimple Number of fome one fuperiour Species, or a mix'd Number of feveral Species, (the Rule for the performing of which, is contain'd in that of the next *Cafe*.) For *Example*, feveral Numbers of *d.* or *fh.* being propofed to be added, we may find the Sum all in *d.* or *fh.* or in *l. s. d.* as far as the Value will reach. So if we take the Shillings of the preceding *Examp.* for the given Numbers of a Queftion, the Anfwer is either 48 r. or 2 *l.* 8 *s.* which you'll find equal to it by the Rule given in the following *Cafe.*

CASE II. *To add* MIX'D NUMBERS.

Rule 1. In every Line of mix'd Numbers let the Species be diftinctly feparated, and in order to this, write firft down the Names (or Characters) of the Species, [the higheft on the Left-hand, and the reft in order toward the Right] and then write every Line of Numbers in order under thefe, the Number of each Species under its own Name; and in every Species obferve duly the Order of places of each Figure.

2. Beginning at the loweft Denomination, add all the Numbers of that Column together (as fimple Numbers) and when you have found the Sum, you muft find how many Units of the next fuperiour Denomination it's equal to: Thus; Divide it by that Number of the Species added, which is equal to an Unit of the next above; what remains in the Divifion write down under the Numbers added, as a Part of the total Sum which belongs to that Species, and the Number of the Quote take and add to the Numbers of the next Species. But if the Sum is lefs than an Unit of the next, fet down what it is, and there is nothing to be carried to the next. Go thus thro' every Denomination, till you come to the laft or higheft, and write down the total Sum of that as it is, becaufe it has relation to no higher; and all thefe Numbers, fet down under every Denomination, make the total Sum.

Examp. of Money:

l.	fh.	d.	f.
4760	14	11	2
6854	16	10	3
5923	08	06	2
640	10	09	0
832	00	11	3
7925	18	08	1
8894	19	10	3
35832	10	08	2

Thus, in the annex'd Example, the total Sum of the Farthings you'll find to be 14; then, becaufe 4 Farthings = 1 Penny, I divide 14 by 4, the Quote is 3, and 2 remains, which is written in the Sum under Farthings, and the Quote 3 I carry to the Pence; and the Sum is 68 *d.* which I divide by 12 (becaufe 12 *d.* = 1 *fh.*) the Quote is 5, and 8 remains, which is written down under *d.* and the 5 carried to the Shillings, whofe Sum is 90, which I divide by 20 (becaufe 20 *fh.* = 1 *l.*) the Quote is 4, and 10 remains, which is written under *fh.* and the 4 carried to the Pounds, whofe Sum is 35832, making the total Sum 35832 *l.* 10 *s.* 08 *d.* 2 *f.*

The *Reafon* of making the Divifions directed (which is the only new thing we have to account for) is plainly this, *viz.* Becaufe 4 *f.* = 1 *d.* therefore as oft as 4 Farthings is contained in any other Number of Farthings, fo many Pence is that Number of Farthings equal to: and the like Reafoning is good in all other Cafes.

But for thofe who do not yet underftand the Rule of Divifion (and even tho' they do) the following is a convenient Method.

To add MIX'D NUMBERS *without the Rule of Divifion.*

Rule. Begin at the lowest Species, and add the Numbers thereof together; not by fingle Columns, as you do fimple Numbers, but take the whole Number that is in every Line together, and add them to one another, pointing when you come to fuch a Sum as is equal to, or greater than an Unit of the next higher Species, (but lefs than two fuch Units) and carry on the Excefs, adding it to the next Number; and fo thro' all that Species; fetting down in the total Sum what Excefs there happens to be after the laft Point. Then for every Point carry 1 to the next Species, and go thro' all the Species in the fame manner; but the higheft you are to add by fingle Figures, as fimple Numbers.

Examp.

l.	s.	d.
46 :	14 :	6
68 :	18 :	11
72 :	10 :	10
94 :	9 :	8
282 :	13 :	11

The Operation of this Example is thus: Beginning at the Pence of the loweft Line, I fay $8 + 10 = 18$, which is $12 + 6$; therefore I make a Point at the 10, and carry forward the 6; thus $6 + 11 = 17$, which is $12 + 5$, this makes another Point at 11, and 5 to carry forward; then $5 + 6 = 11$, which being lefs than 12, I write it down: Then for the two Points I carry 2 to the Shillings; thus, $2 + 9 (= 11) + 10 = 21$, for which I make a Point (for the 20) at 10, and carry forwards the 1 over 20; thus $1 + 18 (= 19) + 14 = 33$, for which I make another Point at 14, and the Remainder 13 (over 20) is written down: then for thefe two Points I carry 2 to the Pounds, and add them as in *Cafe* 1.

Obferve, We need not point the Shillings, but take this eafy Method, *viz.* Add the firft Column and write down what's over 10's, (as fimple Numbers) then carrying the Number of 10's to the fecond Column, (or Place of 10's) fum it up, and if the Sum is an even Number, fet down 0 and carry the half of the Sum to the Pounds; but if it's an odd Number, fet down the odd 1, and carry the half of the Remainder.

The Reafon of this Practice is plain, for two 10's make 20; and we may eafily fuppofe any body, the leaft acquainted with Addition, can take the half of an even Number.

ANOTHER METHOD.

Some propofe to make Tables, that may ferve inftead of Divifion; whereby, when the Sum of any Species is taken by itfelf (as fimple Numbers) you may, by Infpection, find how many Units are to be carried to the next higher Species: The Method of which will be very obvious, by confidering the following *Examp.* for Money, which is made by fimple Addition. Thus; Beginning at 4 *f.* write againft it 1 *fb.* then $4 + 4 = 8$, and againft it write 2; then $8 + 4 = 12$, and fo on, ftill adding the laft Sum to the firft Number. The fame way proceed in all other Species. And for the Length of the Table, you carry it on as far as you pleafe; which, for long Pages or Columns of Numbers may require 40 Lines: But if you'll fubdivide the Column into Parcels of about 12 or 20 Numbers, taking their Sums feparately, and adding them together, then a Table carried fo far will be fufficient.

TABLE *for the Addition of Money.*

f.	d.	d.	fb.	fb.	l.
4 =	1	12 =	1	20 =	1
8 =	2	24 =	2	40 =	2
12 =	3	36 =	3	60 =	3
16 =	4	48 =	4	80 =	4
20 =	5	60 =	5	100 =	5
24 =	6	72 =	6	120 =	6
28 =	7	84 =	7	140 =	7
32 =	8	96 =	8	160 =	8
36 =	9	108 =	9	180 =	9
40 =	10	120 =	10	200 =	10

The

The *Use* of this Table (and all others of the same kind) is obvious; for, having summed any Species, seek that Number in this Table (in its proper Column, if it does not exceed the greatest Number in the Table) and if that precise Number is not there, take the next lesser, and against it you have the Number of the next superiour Species contained in the Sum: which Number you are to carry to the next superiour, and take the Difference betwixt the Sum and that Number next less, which you are to write down under the Numbers added.   I shall leave you to examine the preceding *Examples* by this Table, or make others for your Exercise.

Observe, Those who propose such Tables do it to prevent blotting of Accounts by pointing; for they design them for the more Ignorant, who can't do Division. But as I have said enough already to the Objection against pointing, I shall only observe, that any Accounts such Persons can be intrusted with, can't require so great Nicety as to make pointing a Fault. And I think 'tis plain, there is less Trouble with it in the Practice, and is even more convenient than to do the Work by Division (when one can do it so) because more simple.

EXAMPLES *for the Exercise of ADDITION in* MIX'D NUMBERS.

Money.

20 l.	12 s.	4 d.	f.
346	14	08	3
268	16	10	2
4689	09	11	2
7846	10	06	3
6320	00	04	0
25683	18	00	2
64	12	11	1
45219	17	05	1

Troy Weight.

12 ℔.	20 ℥.	24 dw.	gr.
4768	11	18	20
2345	10	15	23
3689	08	10	18
875	06	12	10
762	04	19	06
86	07	08	09
67	11	13	22
12597	01	19	12

Averdupoife Weight, the Greater.

Ct.	4 qr.	28 ℔.
372	3	27
468	2	20
593	0	10
678	2	18
976	3	19
678	1	24
789	2	06
4558	1	12

Wine Meafure.

Ton.	4 hd.	63 gal.	4 qt.	2 pt.
436	3	62	2	1
678	2	60	3	0
569	1	48	1	1
456	0	29	3	1
789	1	36	2	0
987	2	54	1	1
672	3	46	3	1
4591	1	24	1	1

Long Meafure.

Mile.	8 furl.	220 yd.	3 f.	12 in.
3467	5	219	2	10
4567	7	184	1	11
5678	6	062	2	08
78967	4	009	0	09
56789	3	084	2	11
24608	2	147	1	10
35791	1	210	2	06
209871	0	80	0	05

In this *Example* of *Long Meafure,* becaufe 220 Yards = 1 Furlong, therefore in adding up the Yards, the eafieft way is to add up the Column of Units, writing down what's over 10's, and carrying the Number of 10's to the other two Columns, fum them both together, pointing at every 22, or dividing the Sum by 22.

M   GENERAL

GENERAL SCHOLIUM, *concerning the more special Application of the Rules of Arithmetick.*

A good *Arithmetician* muft be capable of fomething more than barely to perform any Operations with given Numbers, when the Queftion is fimply propofed to Add, Subtract, &c. *i. e.* when he knows what Operation is to be applied, and to what Numbers. For, the great Art of Application lies in the Solution of fuch Queftions as, to diftinguifh them from the other, I call *Mix'd* or *Circumftantiate* Queftions, *i. e.* wherein no Operation is named, but we are left to find the proper Work from the Nature and Circumftances of the Queftion. Now, for this there are not any determinate and general Rules: it depends upon the good Senfe and Judgment of the Arithmetician, whereby he can diftinctly and perfectly comprehend the Nature and Circumftances of a Queftion. It fuppofes him to underftand the Nature of the Subject about which the Queftion is; and laftly, to underftand perfectly well the true general Import and Effect of the feveral fimple Operations of Arithmetick. By which means he may know when the Reafon of the Queftion requires fuch an Operation.

The more fimple the Circumftances of a Queftion are, it will be the more eafy; and where there is but one Operation to be applied, it will be always obvious: But, where a Variety of Circumftances occur, and feveral Operations become thereby neceffary, the Difficulty increafes; which Experience only can make eafy. And therefore, as a proper Introduction to that Experience, I fhall give you, after each of the Rules, fome practical Queftions, whofe Solutions being confidered, may help to guide the Judgment in like Applications.

The Effect of *Addition* being the Difcovery of a Number, which is equal to certain given Numbers, taken all together; whenever the Senfe and Reafon of a Queftion fhews that any given Numbers muft be collected, or that the Number fought is equal in Value to feveral Numbers given, then Addition is the Rule; as in the following Examples.

MIX'D PRACTICAL QUESTIONS *for* ADDITION.

Queft. 1. A Father was 18 Years 4 Months old, (reckoning 13 Months to one Year, and 28 Days to one Month) when his eldeft Child was born. Betwixt the eldeft and fecond were 11 Months 10 Days. Betwixt the fecond and third were 3 Years 8 Months. When the third is 12 Years, 6 Months, 20 Days, how old is the Father? Anfwer 35 Years, 4 Months, 2 Days. For that all thefe Numbers ought to be added together, is manifeft.

	13	28
ye.	mo.	da.
18 :	04 :	00
:	11 :	10
3 :	08 :	00
12 :	06 :	20
35 :	04 :	02

Queft. 2. I bought a Parcel of Goods, whereof the firft Coft was 40 *l.* 10 *s.* paid for packing them 13 *s.* for Carriage 1 *l.* 6 *s.* 8 *d.* and fpent about the Bargain making, 15 *s.* 6 *d.* What do thefe Goods ftand me in all? Anfwer, 43 *l.* 5 *s.* 2 *d.*

l.	s.	d.
40 :	10 :	00
00 :	13 :	00
01 :	06 :	08
00 :	15 :	06
43	05 :	02

Queft.

Queſt. 3. There is owing me by the following Debtors, *viz.*
A. owes 20 *l.* 15 *s.* *B.* owes 100 *l.* *C.* owes 56 *l.* 10 *s.* 8 *d.* *D.* owes
82 *l.* 18 *s.* 4 *d.* What is the Amount of the whole? Anſwer,
260 *l.* 4 *s.*

l.	*s.*	*d.*
20 :	15 :	00
100 :	00 :	00
56 :	10 :	08
82 :	18 :	04
260 :	04 :	00

Queſt. 4. The Diſtance betwixt two Places is ſuch, that if 3
Miles and 5 Furlongs is taken from it, what remains is equal to
8 Miles, 4 Furlongs and 100 Yards. What is the Diſtance of
theſe two Places? Anſwer, 12 Miles, 1 Furl. and 100 Yards.

m.	*furl.*	*yd.*
3 :	05 :	000
8 :	04 :	100
12 :	01 :	100

§. 3. SUBTRACTION *of* APPLICATE NUMBERS.

CASE 1. To ſubtract ſimple Numbers of one Denomination.

Rule. This is done as *Abſtract Numbers.*

	l.
Sub^d.	468
Sub^r.	253
Diff.	215 *l.*

	Gallons.
Sub^d.	72306
Sub^r.	9462
Diff.	62844 *Gallons.*

CASE 2. To ſubtract *Mix'd Numbers.*

Rule 1. Having ſet the Subtractor orderly under the Subtrahend, with a due regard to
the Places and Species; 2. Begin at the loweſt Species of the Subtractor, and take the
Number of that from its correſpondent in the Subtrahend, and write down the Difference.
Do the ſame with all the Species, and you have the Difference ſought.

But if the Number of any inferiour Species in the Subtractor, is greater than its Cor-
reſpondent in the Subtrahend, then to this add a Number equal to an Unit of the next
Species [for *Examp.* to *ſb.* add 20, and to *d.* add 12] and ſubtract from that Sum; and
then to the Number of the next Species in the Subtractor add 1, (or take 1 from the
Subtrahend) and then ſubtract: And when you come to the higheſt Species, do as in
ſimple Numbers, *i. e.* add 10 where you can't ſubtract.

Examp. 1.

l.	*ſb.*	*d.*
Sub^d. 72 :	18 :	10
Sub^r. 29 :	10 :	04
Diff. 43 :	08 :	06

Examp. 2.

l.	*ſb.*	*d.*
48 :	14 :	6
26 :	17 :	10
21 :	16 :	08

The firſt *Example* is ſimple and eaſy. For the ſecond, I do it thus, 10 (in the *d.*)
from 6 cannot, but from 12 + 6 = 18, and 8 remains, to be ſet down in the *d.* Then
1 + 17 (in *ſb.*) = 18, which from 14 cannot, but from 20 + 14 (= 34) and 16 remains,
to be ſet down in *ſb.* Then 1 + 6 (in *l.*) is 7, which taken from 8, 1 remains: Then 2
from 4, and 2 remains. So the total Difference is 21 *l.* 16 *s.* 8 *d.*

The *Reaſon* of this Practice is ſufficiently plain from the like Reaſoning uſed for the
abſtract Rule.

SCHOLIUMS.

1. What was obferved in Addition, is true here alfo, *viz.* That Number ought to be fet down in any inferiour Species, but what's lefs than an Unit of the next; otherwise the Species are confounded: Yet ftill it's arbitrary to make any Species the higheft, and then in it we write down any Number.

2. *Obferve,* That this *Rule* fuppofes we can readily (or in our Mind) difcover the Difference betwixt any Number of any Species lefs than an Unit of the next higher, and any other Number greater, but lefs alfo than fuch an Unit; or betwixt any fuch Number and a leffer increafed by a Number equal to fuch an Unit. And to do this readily, requires a little Practice; but then the Operation for each Species may be performed feparately, Figure by Figure, which will remove this Suppofition; yet it will be more tedious Work; and we muft by Practice acquire the Capacity which the Rule fuppofes. And to make it fomewhat eafier, take this Method: When the Subtrahend Figure is leaft, take the Subtractor Figure out of the Number to be added, and the Remainder add to the Subtrahend Figure; the Sum is the Number to be fet down in the Difference. As in *Examp.* 2. above, I fay 10 *d.* from 6 can't, but from 12, and 2 remains; which added to 6, makes 8 to be fet down.

Other *Examples* in Subtraction.

Troy Weight.				Wine Measure.				Long Measure.			
12	20	24		33	4	2			4	4	
℔.	oz.	dw.	gr.	bd.	gal.	qt.	pt.	yd.	qr.	na.	
342 : 08 : 10 : 06				24 : 42 : 2 : 1				18 : 2 : 0			
84 : 03 : 15 : 20				15 : 56 : 3 : 0				9 : 3 : 1			
258 : 04 : 14 : 10				8 : 48 : 3 : 1				8 : 2 : 3			

MIX'D PRACTICAL QUESTIONS *for* SUBTRACTION.

	l.	*s.*	*d.*
	20	13	4
	8	16	8
	11	16	8

Queft. 1. Having borrowed 20*l.* 13*s.* 4*d.* and paid thereof 8*l.* 16*s.* 8*d.* What's yet due? Anfwer, 11*l.* 16*s.* 8*d.*

	C.	qr.	℔.
	2	3	00
	1	2	14
	1	0	14

Queft. 2. Having bought 2 hund. Weight, and 3 qr. of Sugar; and fold thereof 1 hund. 2 qr. and 14 ℔. what is yet unfold? Anfwer, 1 C. 14 ℔.

	13	28	
	Ye.	mo.	da.
	56	3	22
	24	9	10
	31	7	12

Queft. 3. A Father was 24 Years, 9 Months, 10 Days old when his eldeft Son was born; and is now 56 Years, 3 Months, and and 22 Days. How old is the Son? Anfwer, 31 *ye.* 7 *mo.* 12. *da.*

	l.	*s.*	*d.*
	50	00	00
	36	06	08
	13	13	04

Queft. 4. What is that Sum of Money, which being added to 36*l.* 6*s.* 8*d.* will make 50*l.*? Anfwer, 13*l.* 13*s.* 4*d.*

The

The following two Queſtions are mixed in *Addition* and *Subtraction.*

Queſt. 5. Having borrowed 100 *l.* and paid at one time 20 *l* : 13 *s* : 4 *d.* at another time 33 *l* : 6 *s* : 2 *d.* How much is yet due? *Anſw.* 46 *l* : 6 *d.*

	l : *s* : *d.*
Borr^d	100 : 00 : 00
	20 : 13 : 04
	33 : 06 : 02
Paid	53 : 19 : 06
	46 : 00 : 06

Queſt. 6. There are 2 Casks of Sugar; the Weight of the one full Cask is 1 *C* : 2 *qr.* of the other 1 *C* : 18 *lb.* The Weight of one empty Cask is 25 *lb.* of the other, 1 *qr* : 7 *lb.* What is the Weight of Sugar in both Casks? *Anſw.* 2 *C* : 14 *lb.*

	C : *qr* : *lb.*
	1 : 2 : 00
	1 : 0 : 18
Full Casks	2 : 2 : 18
	0 : 0 : 25
	0 : 1 : 07
Empty Casks	0 : 2 : 04
	2 : 0 : 14

§. 4. MULTIPLICATION *of* APPLICATE NUMBERS.

INTRODUCTION.

IN order to know what Variety there is in *Multiplication of Applicate Numbers,* we muſt firſt conſider, What is a *Simple* and *Proper* Queſtion of Multiplication; and I think it is plain, That this is a Queſtion propoſed with no other Circumſtances, or in no other Form, but barely *To multiply one Number by another;* and then it is certain that the given Numbers muſt be ſuch as are agreable to the general Nature and Definition of that *Operation;* which being no other than the repeating or taking the one Number as oft as the other expreſſes, or contains *Unity;* it is manifeſt, that the Multiplier muſt always be an abſtract Number, expreſſing ſimply the number of times that the Multiplicand is to be taken. Therefore the whole Variety in Multiplication of Applicate Numbers depends upon the *Multiplicand;* which may be either a *ſimple* or *mixed* Number, making only two different Caſes.

Hence we ſee, That the propoſing ſimply to multiply one Number by another, both Applicate, is abſurd. For Example, to multiply 8 *l.* by 3 *l.* For what's that to ſay, *To multiply by 3 l?* To multiply by 3 is intelligible; but what has the Name *l.* to do here? Now, becauſe ſome have thought fit to propoſe ſuch Queſtions without explaining the true Senſe and Meaning of them, and given us very perplex'd Rules for ſolving them, which may be made more general and much eaſier, if the Senſe of the Queſtion is once rightly conceived; I ſhall here therefore make it appear more evidently, that Queſtions propoſed in that ſimple manner, are Nonſenſe in Terms; by which means I ſhall lead you to the Senſe that muſt be put upon all ſuch Queſtions: and in its proper place you'll find the Rules for ſolving them.

Suppoſe then, that it is propoſed to multiply 8 *l.* by 3 *l.* I ſhould ask the Propoſer, What he means by multiplying 8 *l?* He can make no other Anſwer, But that it is the repeating 8 *l.* a certain number of times, (that being the ſimple and proper Definition of *Multiplication.*) Then I ask him, How many times he would have it repeated? And if

he

he anſwers 3 *l.* I hope the Abſurdity is manifeſt. He may indeed ſay, that he means to have it taken as oft as 3 *l.* contains 1 *l.* But then the direct ſimple Anſwer to my Queſtion is 3 times, and not 3 *l.* And ſo the Denomination of *l.* applied to 3, does not belong to it as a Multiplier, and is only a certain circumſtantial way of ſignifying how oft the *Multiplicand* is to be repeated: which is going round about to no purpoſe, when your Meaning can be expreſſed more ſimply; for you ſee it muſt ſtill end in this, that the Multiplier is an abſtract Number expreſſing only how oft the other Number is to be repeated.

Again obſerve, That the Authors of ſuch Queſtions give us this for a Principle, That the two Terms muſt be both applicate to one kind of thing, as *Money, Weight,* or *Time;* ſo that they would very readily pronounce this Queſtion, Nonſenſe, *viz.* To multiply 8 *l.* by 3 Days. But here their confuſed Notion of this matter will appear yet clearer; for this Propoſition is every whit as reaſonable as the other: Becauſe if 3 *l.* cannot be a direct and reaſonable Anſwer to that Queſtion, *How oft*? till you explain it by ſaying, as oft as 3 *l.* contains 1 *l.* it is plain, that by the ſame Method, I may propoſe to multiply 8 *l.* by 3 Days; meaning to take 8 *l.* as oft as 3 Days contain 1 Day; which is equally good Senſe as the other.

But *further:* Since an Explication is neceſſary; and ſomething muſt be underſtood which is not directly expreſſed in propoſing to multiply 8 *l.* by 3 *l.* it may as well ſignify the taking of 8 *l.* 60 times, [*i. e.* as oft as 3 *l.* contains 1 *ſb.* which is 60 times;] for ſo I may explain it. *Again,* to multiply 8 *l.* by 3 *ſb.* ſignifies, according to their meaning, taking $\frac{3}{20}$ Parts of 8 *l.* (becauſe 1 *ſb* $= \frac{1}{20}$ of a *l.*) which is not purely a Queſtion of *Multiplication,* but of that and *Diviſion* too. But why may it not as well ſignify the taking 8 *l.* 3 times (*i. e.* as oft as 3 *ſb.* contains 1 *ſb.*) or 36 times, (*i. e.* as oft as 3 *ſb.* contains 1 *d.*) All theſe meanings are equally reaſonable; ſince the ſimple propoſing to multiply 8 *l.* by 3 *ſb.* limits it to none of them: for it has no meaning till it be explained; or, it has any one of theſe meanings indifferently. And this ſhews how ambiguous ſuch Propoſitions are; or rather, no Propoſitions at all, till their meaning is thus cleared up and determined.

But now, after all this, you are to know that Queſtions may occur which are ſolved by *Multiplication;* yet the mix'd Circumſtances of the Queſtion be ſuch, that all the given Numbers may be particularly applied; whereby the Reaſonableneſs of ſuch Propoſitions as I have here cenſured, may ſeem to be fairly accounted for. To which I anſwer, That in all ſuch Caſes, before we know what is to be done, we are obliged to reaſon upon the Nature and Circumſtances of the Queſtion, and by that means we diſcover, that ſome one given Number of things is to be taken as oft (*i. e.* multiplied) as ſome other contains an *Unit* of a certain Denomination. But then, as the *Multiplication* imports only the taking a number of things a certain number of times, ſo in the Operation, the *Multiplier* ſignifies only *how oft* the other is taken, (this being the proper and formal Notion of a *Multiplier*) tho' it is diſcovered by the Circumſtance of a Number applied to ſome particular thing in the Queſtion.

Again, the very ſame Number of things in different Queſtions will be conſidered in different Views, when it expreſſes the *Multiplier;* and ſo make different *Multiplications;* which ſhews that a ſimple Propoſition of multiplying 8 *l.* by 3 *l.* or 3 *ſb.* (and all Queſtions of this kind) have no determinate Meaning, or are plainly Nonſenſe. Nor is the Propoſition leſs or more reaſonable, tho' the two Terms are applied to different things which can be explained to the ſame meaning as the other, and occur as often in the Circumſtances of mix'd Queſtions.

I ſhall illuſtrate all this by particular Examples immediately; but ſhall firſt explain the Practice in ſimple Queſtions: which, according to the Diſtinction above-mentioned, conſiſts of two Caſes.

CASE

CASE I. *To multiply a* Simple Number.

RULE. This is to be done by the *General Rule for Abstract Numbers* ; and the Product is an Applicate Number of the same Denomination with the *Multiplicand.*

Example 1.	*Example* 2.
7464 *L.*	467 Years.
8	5
59712 *L.*	2335 Y.

SCHOLIUMS:

I. Since any one of two Numbers may be made the Multiplier, and that it is sometimes more convenient to make it the one than the other, tho' that one happens to be the Applicate Number, (*i. e.* the Number to be multiplied) there is no matter; because it is but supposing the Denomination to be shifted and applied to the other Number, and then the Effect is the very same. For Example: To multiply 3 *l.* by 48, I apply 3 as the Multiplier, as if it were 48 *l.* by 3 ; whereby I make the Multiplier according to its proper Notion, Abstract; which gives the same Product and of the same Denomination: For since 48 times 3 = 3 times 48 ; if the Denomination of the Multiplicand is the same, the Product is in all respects the same.

II. *Multiplication* being in effect only a compendious Addition of Numbers equal among themselves, or the Repetition of the same Number, the Observations made in *Schol.* 2. after *Case* 1. in *Addition*, are applicable here also: *viz.* That the Product of a mixed Number may be found and expressed two ways; either, 1. By taking the Product of each Number by itself as so many distinct Questions and Applications of *Case* 1. which make a confused Answer, tho' the Work is simple Multiplication. Or, 2. Regarding the Relations of the Species, and expressing it in the highest as far as possible ; which is considering it properly as a mixed Number, whereby we make a more simple and useful Answer, tho' the Operation is more complex; for it requires Division, the same way as Addition of mixed Numbers.

III. *Again* ; When the Number multiplied is a simple Number of such things as have a superiour Species, the Product may be found also simply in its own Species; or may be expressed in superiour Species, as far as it reaches; according to the Method of the following Rule for mixed Numbers. But whereas in Addition we had an easy way to supply the Rule of Division by pointing, we cannot apply that Method here; and therefore Division is indispensible, if we would express the Product in its highest Species, unless we make use of the Tables described already for Addition; and then the Examples must be either very small Numbers, or the Tables exceeding large. But there is yet another way of expressing the Product of mixed Numbers simply without Division, which you'll find in the next Case. In the mean time, before we proceed to that, there is a special Class of mixed Practical Questions preparatory to it, which are Applications of the first *Case*, comprehended under this Title ; *viz.*

REDUCTION *from a* Higher *to a* Lower *Species* ; i. e.
Finding a Number of things of a Lower Species, (Denomination, or Value) Equivalent to a given Number of a Higher Species. For which, this is the Rule.

RULE.

R U L E. Multiply the given Number by that Number of the Species to which you would reduce it, which makes an *Unit* of the Species reduced; the Product is the Number-fought.

Examp. 1.	*Examp.* 2.
48 *l.*	34 ℔.
20	12
960 *fb.*	408 *oz.*
12	20
11520 *d.*	8168 *dw.*
4	24
46080 *f.*	196032 *gr.*

Exam. 1. To reduce 48 *l.* to *fb.* it is $= 960$ *fb* and this again to *d.* is $= 11520$ *d.* and again to *far.* is $= 46080$ *f.* According to the Operation in the Margin; each Step being in effect a new Question.

Examp. 2. To reduce 34 ℔ *Troy* Weight; it is $= 408$ *oz* $= 8168$ *dw* $= 196032$ *gr.*

The *Reason* of this Practice is obvious: For if 1 *l.* $= 20$ *fb.* then 48 *l.* is $= 48$ times 20 *fb.* or 20 times 48 *fb.* which is the same thing. (The like Reason will be found in other Cases.) Therefore you are to *observe*, That tho' the Denomination of *l.* is left with the given Number 48, as it stands in the Operation, yet it is performed in a quite different view; for we are to confider it not as the Multiplication of 48 *l.* by 20, the fimple effect of which would certainly be, 960 *l.* But from the Reasons now explained, we confider it as the Multiplication of 48 *fb.* by 20, or of 20 *fb.* by 48; which is $= 960$ *fb.* And fo of the reft. Which takes away the feeming Abfurdity of naming the Product differently from the *Multiplicand.*

SCHOLIUMS.

I. If it is required to reduce a Number to a Species which is not immediately the next to it, as *l.* to *d.* We may either do it by Steps thro' all the intermediate Species, as above; or it may be done at one Multiplication, if we know how many *Units* of that Lower makes 1 of the Higher. For Example; To reduce *l.* to *d.* we multiply by 240; becaufe 1 *l.* $= 240$ *d.* And for this purpofe, it is ordinary to have Tables of Reduction, fhewing how many *Units* of any Species (of common Ufe in Bufinefs) make 1 of any other; which are eafily made by the preceding Rule, from the known Relations betwixt each Species and the next, which you have in the Tables of *Addition.* One or two Tables will be fufficient here to explain this Matter; and you may make the like for all other mixed Numbers, at your pleafure.

T A B L E S *of* R E D U C T I O N.

1. *Money.*				2. *Troy Weight.*			
l. :	*fb* :	*d.* :	*f.*	℔ :	*oz* :	*dw* :	*gr.*
1 =	20 =	240 =	960	1 =	12 =	240 =	5760
	1 =	12 =	48		1 =	20 =	480
		1 =	4			1 =	24

The Ufe of thefe Tables is plain; for under every Denomination you have 1, and in the fame Line the Number to which this 1 is equal of each inferiour Species: So, for Example, to reduce *l.* to *f.* at once, the Multiplier is 960.

But you'll find that it will generally be as convenient to reduce any given Number to any inferiour Species by reducing it gradually thro' all the Denominations, efpecially for

this

this Reafon, that it were impoffible to remember all thefe Tables, and too much trouble to turn to them upon every occafion; whereas the gradual Relations of the feveral Species are eafily kept in mind. And laftly, the Reductions that moft frequently happen, are of mixed Numbers, which neceffarily muft be done gradually; as in the following Article.

II. If a *mixed Number* is propofed to be reduced to the loweft Species expreffed in it, [or to the loweft poffible;] begin at the higheft, and reduce it to the next Species, adding to the Product the given Number of that Species. Then reduce this Sum to the next Species, and fo on thro' them all, taking in always the given Number of every inferiour Species: As in the following Example; which needs, I think, no further Explanation.

Examp. 1.

l	:	*lb*	:	*d*	:	*f.*
724	:	17	:	09	:	2

20

14480 *lb.*
17 add

14497 *lb.*
12

173964 *d.*
9 add

173973 *d.*
4

695892 *f.*
2 add

695894 *f.*

Obferve, The Numbers of the inferiour Species may be taken into the Product of their Species without the pains of writing them down and adding them, by adding them Figure by Figure to the like places of the Product, as they are found in the Multiplying. You'll eafily underftand the Method by examining the following Examples. I fhall only further obferve, that it is beft to take them all in upon the Multiplication with the Units place of the Multiplier, in cafe when this has two Places, you do the Work at length. But in the following Examples I have made the Product at once.

Ex. 2. *Troy Weight.*

lb	·	*oz*	:	*dw*	:	*gr.*
48	:	09	:	16	:	23

12

585 *oz.*
20

11716 *dw.*
24

281207 *gr.*

N. B. In this *Ex.* 2. as you may reduce the *dw.* at two Steps; multiplying firft by 6, and then by 4, (becaufe 6 × 4 = 24) you muft mind that the 23 *gr.* are to be taken in with the laft Multiplication.

Ex. 3. *Averdupoife Weight.*

St	:	*lb*	:	*oz*	:	*dr*	:	*qr.*
256	:	12	:	14	:	08	:	2

16

4108 *lb.*
16

65842 *oz.*
16

1052474 *qr.*

Thefe Examples fhew the Practice fufficiently; and we need give no other, but leave Examples of other kinds of Things to the Student's own Choice and Exercife. You'll obferve here alfo the great Difference betwixt multiplying a *mixed Number,* and reducing it; tho' this is performed by Multiplication: For multiplying it, is a Repetition of the Whole fo many times, or finds a Number which contains every Part of the given mixed Number fo many times; but reducing is only finding a Number equal to the given mixed Number in the loweft Denomination; in which every part of it is differently multiplied, and the laft part not multiplied at all. So in the firft Example, 724 *l.* is multiplied by 20, and 12, and 4 continually, that is by 960. But 17 *lb.* is only multiplied by 12 and 4, or 48. and 9 *d.* only by 4. The Anfwer of theQueftion being 695894 *f.* Whereas the Product of 724 *l.* : 17 *s* : 9 *d* : 2 *f.* by any Number would be equal to that Number of times 695894 *f.*

N

CASE

CASE II. *To multiply a* Mixed Number.

RULE. Begin at the loweſt Species of the *Multiplicand,* and having multiplied that Number, reduce the Product to the next Species; *i. e.* find by Diviſion [in the manner already explained in *Addition of Mixed Numbers*] how many Units of the next ſuperiour Species it is equal to, and what remains over; ſet what is over as a Part of the Anſwer of the Denomination multiplied. Then multiply the given Number of the next ſuperiour Species, and to the Product add that Number to which the Product of the preceding Species was reduced; and reduce this Sum to the next ſuperiour Species; marking the Remainder, or what is over, as a Part of the Anſwer of the Species multiplied; and go on thus thro' all the Species of the *Multiplicand.*

Examp. To multiply 236 *l* : 14 *ſb* : 9 *d.* by 26.

In the annex'd Scheme you ſee the Method of the Work of this Rule; except that the Effect of the Reduction of the ſeveral Products is ſet down without the Operation by which it was done; theſe being ſuppoſed to be done a-part by themſelves, and transferred to this Scheme. But you'll find another way immediately, wherein Diviſion is uſed, and the whole Operation appears in one Scheme, without any confuſion.

```
   l  :  ſb  :  d.
 236  :  14  :  09
                26
 ─────────────────────
6136  :  364  :  234  Prod. of each Species.
         19  :   6 . . = 234 d.
         383 . . . . . = 364 + 19 ſb.
    19 :  3  . . . . . = 383 ſb.
 ─────────────────────
6155  . . . . . . . . = 6136 + 19 l.
 ─────────────────────
6155  :  03  :  06  Total Product.
```

Another Method without Diviſion.

Reduce the Multiplicand to the loweſt Species, as has been already taught; then multiply this Number by the given Multiplier, and the Product is the Number of that Species equivalent to the propoſed Number of times the Multiplicand.

```
  Operation.
   l  :  ſb  :  d.
  48  :  16  :  8
           20
 ──────────
         976 ſb.
          12
 ──────────
       11720 d.
          42
 ──────────
      492240 d.
```

Examp. To multiply 48 *l* : 16 *ſb* : 8 *d.* by 42, it is equal to 492240 *d.* as in the Margin.

This Anſwer is the only proper and natural Effect of Multiplication. And if it is required to know the Value of it in higher Species, this is properly a Queſtion of Diviſion, to be performed in the manner already explained; which is to divide by the Number of every Species which makes an Unit of the next above. But I ſhall refer you to Diviſion to ſee the beſt and neateſt Method of ordering the Operation. And here only obſerve theſe two things.

1. That with large Multipliers, this laſt Method (*viz.* of reducing to the loweſt Species by Multiplication, and then to the higher by Diviſion) will generally prove a more convenient Method than that of the firſt Rule. But, 2. If the Multiplier is a ſingle Digit, or any Product of two Digits, the Work may, in moſt Caſes, be eaſily performed according to the firſt Rule, without writing the Diviſions: As in the following Examples.

l : $\text{\it fb}$: $d.$

Ex. 1.   68 : 14 : 09 : 3
                  7
     ———————————————
       483 : 03 : 8 : 1

C : qr : $\text{\it fb}$

Ex. 2.   37 : 3 : 18 by 28.
                4
    ———————————————
     148 : 2 : 16
              7
    ———————————————
     1040 : 2 : 00

Here I fay, $3 \times 7 = 21$; which is 1 $f.$ over 20, or 5 times 4: therefore I write down 1, and carry 5. Then $7 \times 9 = 63$, and 5 carried, is 68 d; which is 8 over 60, or 5 times 12: fo I write 8 $d.$ and carry 5. Then $7 \times 14 = 98$, and 5 is 103; for which I write 3, and carry 5, (for 5 times 20 is 100.) Then multiply the 68 $l.$ and add the 5 from the Shillings.

For the 2d *Examp.* I refolve the Multiplier into 4 and 7. Then beginning with 4; I fay, $4 \times 18 = 72$, which is 2 times 28 ($= 56$,) and 16 over; or refolving the 4, I fay $2 \times 18 = 36$, which is 28, and 8 over; confequently in 2 times 36 ($= 72$) there are 2 times 28, and 2 times 8, or 16 over. The reft is eafy. Then for the 7, I confider that 7 being the 4th Part of 28, and 4 the 4th Part of 16; therefore 7×16, or 16 times 7, is equal to 4 times 28; for which I write 00, and carry 4. Then $7 \times 2 = 14$, and 4 carried is 18; for which I write 2 $qr.$ and carry 4 $C.$

By fuch means as thefe, one may by Practice eafily perform any Queftions of this kind.

SCHOL. As to the Solution of other mixed Queftions, there is no other general Direction can be given whereby to know when Multiplication is to be applied, but only this, *viz.* To confider, that the true Effect of Multiplication being the repeating of any Number, or taking it a certain number of times; therefore whenever the Senfe and Reafon of a Queftion requires that any given Number of things be repeated, or that a Number be found equal in Value to a certain given Number of things, repeated or taken as oft as fome other given Number in the Queftion contains Unity; then Multiplication is the Work required. As in the following Examples.

Mixed Practical Queftions in Multiplication.

Queft. 1. There is in each of 28 Bags, 44 l : 16 $\text{\it fb}$: 8 $d.$ How much is in the Whole? *Anfw.* 1255 l : 6 $\text{\it fb}$: 8 $d.$

l : $\text{\it fb}$: $d.$
44 : 16 : 8
        4
————————————
179 : 06 : 8
        7
————————————
1255 : 06 : 8

Here the Nature of the Queftion plainly requires that 44 $l.$: 16 $\text{\it fb}$: 8 $d.$ be multiplied by 28, the Number of Bags; for if 1 Bag contains fo much, 28 Bags muft contain 28 times fo much, which imports a Multiplication by 28; which is taken abftractly in the Operation, tho' it is applied to Bags in the mixed Propofition. As to the Manner of working, I have chofen 4 and 7 as *Factors;* becaufe $4 \times 7 = 28$.

Queft. 2. At 3 l : 6 $\text{\it fb}$: 4 $d.$ *per* Yard, what is the Value of 465 Yards? The Reafon of this Queftion fhews, that 3 l : 6 $\text{\it fb}$: 4 $d.$ muft be taken 465 times; or multiplied by 465. For the Value of 465 Yards muft be 465 times as much as the Value of 1 Yard. And to do the Work, I reduce 3 l : 6 $\text{\it fb}$: 4 $d.$ to $d.$ it is $= 796$ $d.$ which multiplied by 465, produces 370140 $d.$ which is again equal to 1542 l : 5 $\text{\it fb}$. by Divifion. As you will learn afterwards.

Queft.

$l : \text{\it fb} : d.$
16 : 6 : 8
          6
_____
98 : 0 : 0
18
_____
1764
7
_____
12348

Queft. 3. There are 7 Chefts of Drawers; in each of which are 18 Drawers; and in each of thefe are 6 Divifions; in each of which there is 16 l : 6 *fb* : 8 d. How much Money is in the Whole? *Anfw.* 12348 l.

It is plain there muft be a Sum of Money equal to the continual Product of 16 l : 6 *fb* : 8 d. by 6, 18, and 7.

Queft. 4. If 1 l. give 4 *fb*. of Intereft in any time; How much will 346 l. give in the fame time? *Anfw.* 1384 *fb* = 346 × 4 *fb*. Here the 346 is applied to l. in the Propofition, but is an abftract Number in the Operation; which is not multiplying 4 *fb*. by 346 l. but by the abftract Number 346.

If this Queftion be propofed, *viz.* If 1 *fb*. yield 4 *fb*. Intereft; How much will 4 l : 10 *fb*. yield in the fame time? It is plain it muft be 4 *fb*. taken as oft as 4 l : 10 *fb*. contains 1 *fb*. *viz.* 90 times, (for 4 l : 10 *fb* = 90 *fb*.) which makes 360 *fb* = 90 × 4 *fb*. But this is not multiplying 4 *fb*. by 4 l : 10 *fb*. which would be an abfurd Propofition.

Again, Suppofe the Queftion were; If 1 l. yield 4 *fb*. what will 4 l : 10 *fb*. yield? The Anfwer is 6 *fb*. equal to 4 *fb*. taken as oft as 4 l : 10 *fb*. contains 1 l. which is one and a half. But this, and all other Queftions where Fractions come in, are not fimple Queftions of Multiplication. And as either of thefe Queftions have an equal right to be called the Multiplication of 4 s. by 4 l : 10 s. it fhews us how unreafonable fuch Propofitions are, fince it is the mixed Circumftances of the Queftion that determine how the Multiplication is to be made, which is different in different Circumftances.

§. 5. DIVISION *of* WHOLE *and* APPLICATE NUMBERS.

INTRODUCTION.

BEfore we enter upon this Application, we muft confider the various Senfes that may be put upon *Divifion.*

In the Definition, *Chap.* 6. there is but one Senfe expreffed; but there are other three ways of propofing a Queftion in *Divifion,* fo dependent upon that in the Definition, that the fame Number folves the Queftion in all the Senfes in which it is a poffible Queftion.

The firft Senfe is that in the Definition, *viz.* To find how oft one Number is contained in another. The fecond is to find, What Part of the Dividend the Divifor is equal to. The third is, To find a Number which is contained in the Dividend as oft as the Divifor expreffes. The fourth is, To find a Number which is fuch a Part of the Dividend as the Divifor expreffes or denominates.

Now it will eafily appear, That the Anfwers to all thefe Queftions, or the Impoffibility of fome of them in fome Cafes, is difcovered by dividing according to the preceding Rule taken in the firft Senfe: Thus,

1. Let us firft fuppofe, That the Divifion is without a Remainder, and all is plain: For the Number fhewing how oft one Number is contained in another, (which is the firft Senfe,) does, from the nature of an *aliquot* Part, denominate what Part the Divifor is of the Dividend, (which is the fecond Senfe.) Again, the fame Quote is a Number contained in the Dividend at oft as the Divifor expreffes; (which is the third Senfe,) as has been fhewn in the *Proof of Divifion.* And hence, laftly, it is fuch a Part of the Dividend as the Divifor denominates; (which is the fourth Senfe.)

Examp.

Examp. 12 ÷ 3 = 4, and nothing remaining; that is, 3 is contained 4 times in 12, (the firſt Senſe.) And it is the 4th Part of 12, (the ſecond ſenſe.) Again, 4 is a Number contained 3 times in 12, (the third Senſe.) And it is the 3^d Part of 12, (the fourth Senſe.)

2. Suppoſe in the next place, That the Diviſion has a Remainder; or, that the Diviſor is not an *Aliquot* Part of the Dividend, [which includes that Caſe wherein the Diviſor is greater than the Dividend.] Then the Queſtion is Poſſible or Impoſſible, accoŕding to different Views and Limitations; as I ſhall here explain.

Let the Diviſor be 3, and the Dividend 14; the Queſtion is poſſible in the firſt Senſe ; and the Anſwer is 4 times, if we confine it to the number of times that the whole Diviſor is contained in the Dividend : but taking it in a larger Senſe, the Anſwer is $4 + \frac{2}{3}$. And in this View the Diviſor may be greater than the Dividend : So if we ask how oft 14 is contained in 3, the Anſwer is $\frac{3}{14}$ Parts of a Time; the plain Senſe of which is, that 3 contains $\frac{3}{14}$ Parts of 14.

In the ſecond Senſe, the Queſtion ſuppoſes the Diviſor is an *Aliquot* Part of the Dividend ; and is therefore impoſſible when it is not ſo. But if we take a Part more largely for Part *Aliquot* or *Aliquant,* and ask what Fraćtion the Diviſor is of the Dividend, then the Queſtion is poſſible. But there is no new Queſtion, ſtrićtly ſpeaking ; for it coincides with the firſt Queſtion, changing the Dividend and Diviſor, and taking *how oft* in the largeſt Senſe. Thus ; if we ask what Fraćtion 3 is of 14, the Anſwer is $\frac{3}{14}$; which is alſo the Anſwer, if we ask *how oft* 14 is contained in 3. And therefore I had rather in this Caſe call 3 the Dividend, and reduce it to the former Caſe ; eſpecially for this Reaſon, That the ſame Quote may be the Anſwer to all the really different Senſes of the Queſtion, while the Names of Diviſor and Dividend are applied to the ſame Numbers.

In the third Senſe, If we ask what is the greateſt Number contained 3 times in 14; then if we limit it to a Whole Number, the Anſwer is 4. And if the Diviſor is greater than the Dividend, the Queſtion under this Limitation is impoſſible ; as it alſo is if we ſhould ask, what Whole Number is contained without a Remainder 3 times in 14 ; for this is contrary to ſuppoſition. But if we enlarge the Senſe of the Queſtion, and ask what Number, Whole, or Fraćtion, or Mix'd, is contained in the Dividend preciſely as oft as the Diviſor expreſſes, (ſo that the Produćt of the Diviſor and Quote is equal to the Dividend,) the Queſtion is always poſſible. Thus, as 3 is contained in 14, $4\frac{2}{3}$ times ; ſo $4\frac{2}{3}$ is contained 3 times in 14 ; that is, 4 times any thing, and $\frac{2}{3}$ Parts of that thing is contained 3 times in 14 of that thing. For it has been already ſhewn, (in *Chap.* 6.

Schol. 4. after the Definition of *Diviſion,*) that B times $\frac{A}{B}$ Parts of any thing, is equal to $\frac{A}{B}$ Parts of B times that thing; therefore as 3 is contained in 14, $\frac{14}{3}$ times ; ſo $\frac{14}{3}$ is contained 3 times in 14. Or, becauſe $\frac{14}{3} = 4\frac{2}{3}$; therefore as 3 is contained $4\frac{2}{3}$ times in 14, ſo $4\frac{2}{3}$ is contained 3 times in 14.

In the fourth Senſe, the Queſtion is impoſſible in Pure and Integral Numbers, when the Diviſor is not an *Aliquot* Part of the Dividend; ſo becauſe 3 is not an *Aliquot* Part of 14, there is no Number which is a third Part of 14; for if there were, 3 would be ſuch a Part of 14 as that Number expreſſes. But taking the Queſtion more largely, and admitting Fraćtions, it is in all Caſes poſſible : So $4\frac{2}{3}$ Parts of any thing is a 3d Part of 14, becauſe it is contained in it 3 times.

As to the Circumſtance which makes the third and fourth Senſe poſſible, when the Diviſor is not an *Aliquot* Part of the Dividend ; it is remarkable, That the Subjećt of the Queſtion is not pure Numbers, but ſuch Quantitys expreſſed by Numbers as are diviſible, either really or imaginarily, into Parts or leſſer Quantitys : for in pure Numbers 14 has not a third Part ; but conſidering the 14 as applied to ſome diviſible Subjećt, the Quantity expreſſed by 14 has a third Part, which is expreſſed by $4\frac{2}{3}$; therefore the Queſtion is poſſible only in Applicate Numbers.

From

From what's explain'd we fee evidently, that as all the four Queftions or Views of Divifion, are poffible when the Divifor is an Aliquot Part of the Dividend; fo when it is not an Aliquot Part, there are but three really different Queftions; and which are all poffible when the Subject of the Queftion is not pure Numbers, *i. e.* When we admit another Confideration than that of the Number of Things expreffed, *viz.* their Divifibility into Parts or leffer Quantities: For then a Fraction comes naturally into the Anfwer, and makes a complear Quote.

Now, from thefe different Views and Senfes of Divifion, we learn what Variety there can poffibly be in the particular Application of Numbers for a Queftion of Divifion; of which there can only be two *Cafes.*

1. To make a Queftion in the firft or fecond Senfe, the Divifor and Dividend muft both be applicate, and both to things of the fame kind: And mutually, if the Divifor and Dividend are fo applied, the Queftion admits only the firft or fecond Senfe; and the Quote is an abftract Number, fhewing how oft the Divifor is contained in the Dividend, or denominates what Part the Divifor is of the Dividend, if there is no Remainder. For, as it is a reafonable Queftion to ask, How oft one Number of any kind of Things is contained in, or what Part it is of another of the fame kind of Thing; fo a Queftion being propofed in this manner, and either Divifor or Dividend being applied to a particular kind of Thing, the Nature of the Queftion imports, that the other is alfo applied, and to the fame kind of Thing; fince it's abfurd to ask, How oft a Number of one kind of Thing, as 3 Pounds, is contained in a Number of another kind of Thing, as 14 Days?

2. To make a Queftion in the third or fourth Senfe, the Dividend muft be an Applicate Number, and the Divifor Abftract, denominating what Part of the Dividend the Quote is, or how oft it is contained in it: fo that the Quote is a Number applicate to the fame kind of Thing as the Dividend; the Part of any Thing being of the fame Nature as the whole. Again, mutually the Dividend being confidered as applied, and the Divifor as abftract, the Queftion can admit only of the third or fourth Senfe.

That the Application muft be ordered in the manner now explained, may be alfo deduced from the Connection and Dependence of Multiplication and Divifion: For, fince in Multiplication the Product and one Factor muft be applicate to the fame kind of Thing, the other Factor being abftract; and in Divifion the Divifor and Quote produce the Dividend: it follows, that the Dividend, with the Divifor or Quote, are alike applied, the other being abftract.

Again, *Obferve,* That in mix'd Queftions it will happen that two Numbers which in the Propofition are applied to different Things, muft be divided one by the other: But in this Cafe, you'll always find that the Number made Divifor is confidered in the Operation, as abftract, denominating what part of the other the Nature and Reafon of the Queftion requires to be taken. So that in all Cafes it's true, that the Divifor is either abftract, or applied to the fame kind of thing as the Dividend.

We fhall next explain the fimple Practice in Divifion of *Applicate Numbers.*

CASE 1. The Divifor and Dividend being both applied to the fame kind of Thing.

Rule. Reduce, (if need be) the Divifor and Dividend to fimple Numbers of one Name, (the loweft expreffed in either Term;) then divide thefe Numbers by the general Rule. The Quote fhews how oft the Divifor is contained in the Dividend, or what Part the Divifor is of the Dividend, when there is no Remainder.

Examp. 1.   3 *l.*) 24 *l.* (8   *Examp.* 2.   3 *l.*) 20 *l.* (6⅔

Examp. 3.   3 *lb.*) 12 *l.*
           or,   *years, mo. ye. mo. da.*
           3 *lb.*) 240 *lb.* (80   *Examp.* 4.   3 : 4) 24 : 7 : 18
                                        By Reduction at 13 Months to
Examp. 5.   13 *oz.* 8 *dr.*) 128 ℔.   a Year, and 28 days to 1 Month.
           By Reduction,   1204 *days*) 8950 *da.* (7 $\frac{522}{1204}$
   216 *dr.*) 30208 *dr.* (139 $\frac{184}{216}$ 8428
           216   ――――
           ――――   522
           860
           648
           ――――
           2128
           1944
           ――――
            184

The *Reason* of this *Rule* is plain; for the Divisor and Dividend expressing Things of the same Value and Name, it is evident the Operation is to be managed as with abstract Numbers, by the General *Rule*; *i. e.* The Quantity expressed by the Divisor is contained in the Quantity expressed by the Dividend, as oft as the Divisor is contained in the Dividend, taken purely as Numbers. *Again,* If the Divisor and Dividend express Things of the same general Nature, which can be said to contain one another, then tho' they are not of one particular Species or Name, yet the Question is possible, only it requires that they be reduced to Numbers that express Quantities of one Species or Name; and then it is manifest, that the Division of these Numbers by the General *Rule,* gives the true Quote. So in *Exam.* 1. 3 *l.* is contained in 24 *l.* as oft as 3 in 24. But in *Examp.* 2. 3 *s.* is oftner contained in 12 *l.* than 3 in 12; for it is as oft as 3 in 240, the Number of Shillings equal to 12 *l.* And, because the Divisor and Dividend are then only in a State to be managed as pure Numbers, when they are both simple Numbers of one Name, this shews the Reason of reducing mix'd Numbers.

C A S E 2. The Dividend being *Applicate,* and the Divisor *Abstract.*

Rule. If the Dividend is a simple Number, greater than the Divisor, divide it by the General Rule; the Quote is a Number of the same things as the Dividend : and if there is no Remainder, the Operation is finished; but if there is a Remainder, reduce it to the next Denomination, and divide; and so on, as long as there is a Remainder, and any lower Denomination, and make a Fraction of the last Remainder. Thus you have the Answer in one Species or several; which is an Applicate Number contained in the Dividend as oft as the Divisor expresses, or is such a part of the Dividend as the Divisor denominates. (See *Ex.* 1.)

Again, If the Dividend is a simple Number, less than the Divisor, you must first reduce it to a lower Species, till it be equal to, or greater than the Divisor, and then divide and proceed with the Remainder as before. (See *Ex.* 3, 4.) If it's not equal to the Divisor in any Species, then the Answer is a Fraction of the given Species, whose Numerator is the given Dividend. (*Ex.* 6.)

Lastly, If the Dividend is a mix'd Number, you may do the Work two ways: Either (1.) Reduce it to a simple Number of the lowest Species, and then divide; so you'll have the Answer in that Species, (which may be reduced again to superiour Species by Division, as has been formerly explained, and will be more particularly by and by.) But it will be better to proceed in this manner : (2.) Begin with the Number of the highest Species in the Dividend; divide it, and reduce the Remainder to the next Species, taking in the given Number of that next Species; then divide; and so go on. (See *Ex.* 2.) But if the Number of the highest Species is less than the Divisor, reduce it, taking in the given Number of the next Species, and so on, till you have a Number equal to, or
greater

greater than the Divifor: (*Ex.* 5.) And if that be not in any of the known Species, then the Anfwer is only a Fraction, whofe Numerator is the Dividend, reduced to the loweft Species, and refers to an Unit of that Species. (See *Examp.* 7.)

Examp. 1.

$3\,)\ 14\,l.\ (4\,l.$
12
———
2 Rem.
20 ·
———
40 *s*. (13 *s*.
39
———
1 Rem.
12
———
12 *d*. (4 *d*.

Examp. 2.   *l.*   *s.*   *d.*

$6\,)\ 23:10:8\,(3\,l.$
18
———
5 Rem.
20
———
110 *s* (16 *s*.
108
———
2 Rem.
12
———
32 *d*. (5 *d*.
30
———
2 Rem.
4
———
8 *f*. $(1\frac{2}{3}f.$

Examp. 3.

$24\,)\,18\,l.$
20
———
360 *s*. (15 *s*.
24
———
120
120

Examp. 4.

$42\,)\ 16\,l.$
20
———
320 *s*. (7 *s*.
294
———
26 Rem.
12
———
312 *d*. (7 *d*.
294
———
18 Rem.
4
———
72 *f*. $(1\frac{30}{42}f.$
42
———
30

Examp. 5.   *l.*   *s.*

$14\,)\ 8:15$
20
———
175 *s*. (12 *s*.
168
———
7 Rem.
12
———
84 *d*. (6 *d*.
84

Examp. 6.

$3460)\ 3\,l.\ (\frac{1}{1440}$ of 1 *l.* becaufe 3 reduced to Farthings is but 2880 *f*.

Examp. 7.

$3460)\ 3\,l.\ 12\,s.$
20
———
$72\,(\frac{72}{1440}$ of 1 *s*.

The *Reafon* of this Practice is plain: For, if we find any propofed Part of all the Members of which any Number or Quantity is compofed, we have the like Part of the Whole; fince the Whole is nothing elfe but all the Parts. And if the Dividend is lefs than the Divifor, yet if reduced to another Species, it is equal to, or greater than the Divifor; it's plain that the equivalent Number in another Species being divided by the fame Divifor, gives the true Anfwer: Thus $18\,l.=360\,s.$ therefore the 24th part of 18 *l.* is the 24th Part of 360 *s.*

SCHOLIUM. What's already done fhews the Practice of Divifion, or the Solution of fimple Queftions, where the Propofition is directly and fimply to divide one Number by another, (in any of the Senfes above explained.) And as to the Solution of mix'd Queftions, all the further Direction can be given for knowing when Divifion is to be applied, is to *confider* well the Effect of Divifion as above explained: Which may be reduced to two principal Views, *viz.* Finding how oft one Number is contained in another, or finding a propofed aliquot Part of a Number. Then, when the Reafon and Nature of a

Queftion

Queſtion requires, that you find how oft one Number, ſimple or mix'd, of any kind of Thing, is contained in another Number of the ſame kind of thing; or, that you find ſuch a Part of any Number of things, as another Number denominates, or as an Unit of any Species of Things is of a certain given Number of the ſame Things; then is Diviſion your Work, as in the following Examples.

MIX'D PRACTICAL QUESTIONS *for* DIVISION.

A ſpecial Claſs of ſuch Queſtions is comprehended under Title of,

REDUCTION *from a lower to a higher Species,* i. e. *To find a Number of things of a higher Species, equal in Value to a given Number of a lower Species; or, at leaſt, to find the greateſt Number of the higher contained in the Number of the lower, with what remains over of that lower Species: Suppoſing always, that an Unit of the lower Species is an aliquot Part of an Unit of the higher. For which this is the*

Rule. Divide the given Number by that Number which expreſſes how many Units of the Species to be reduced, are contained in an Unit of the Species to which it is to be reduced. The Quote is the Number ſought of that higher Species, and the Remainder is a Number of the Species reduced.

Thus you may reduce gradually from the loweſt to the higheſt Species; or all at once to the higheſt, if the Number of Units of the loweſt, which make one of the higheſt, is known. (As you may know by the Reduction Tables, explained in §. 4.)

In all the following Queſtions, I have performed the Diviſions by the contracted Methods, explained in *Chap.* vi. §. 2.

Examp. 1. In Money.

4 | 74608395 *f.*
12 | 18652098 *d.* : 3 *f.*
20 | 1554341 *s.* : 6 *d.*
| 77717 *l.* : 1 *s.*

Here 74608395 *f.* being divided by 4, Quote 18652098 *d.* and 3 *f.* over; theſe *d.* divided by 12, Quote 1554341 *s.* and 6 *d.* over; theſe *ſh.* divided by 20, Quote 77717 *l.* and 1 *s.* over. Wherefore 74608395 *f.* = 18652098 *d.* 3 *f.* = 1554341 *s.* 6 *d.* 3 *f.* = 77717 *l.* 1 *s.* 6 *d.* 3 *f.*

By this you know how to explain the following Examples.

Examp. 2. Troy Weight.

24 { 4 | 3476589426 *grains.*
    { 6 | 8691473 56 —
20 | 144857892 *dw.* : 18 *gr.*
12 | 7242894 *oz.* : 12 *dw.*
| 603574 *℔.* : 6 *oz.*

Examp. 3. Long Meaſure.

12 | 46320689372589 *inches.*
3 | 3860057447715 *feet* : 9 *inch.*
220 { 11 | 1286685815905 *yd.* : 00
     { 20 | 116971437809 ½
8 | 5848571890 *furl.* : 105 *yd.*
| 731071486 *miles* : 2 *furl.*

The *Reaſon* of Dividing in theſe Caſes is obvious: For *Ex.* Since 4 Farthings = 1 *d.* therefore as many times as 4 *f.* are contained in any Number of *f.* ſo many *d.* is that Number of *f.* equal to. In which, obſerve, that the immediate Effect of the Diviſion is an abſtract Number, ſhewing how oft 4 *f.* is contained in a greater Number of *f.* and we apply the Name of *d.* to the Quote, from the Reaſon of the Queſtion, as now explain'd.

Queſt. 1. 8 Men have equal Shares of a Stock of 146 *l.* 16 *s.* what is each Man's Share ? Anſwer, 18 *l.* 7 *s.* *viz.* the 8th Part of 146 *l.* 16 *s.* For the Nature of the Queſtion plainly directs us to take an 8th Part. Where obſerve, that tho' 8 expreſſes a Number of Men in the mixt Propoſition, yet in the Operation it is conſidered abſtractly as the Denomination of that Part of 146 *l.* 16 *s.* which the Nature of the Queſtion requires to be taken.

8) 146 *l.* : 16 *s.*
18 : 7

Queſt.

Quest. 2.

y. w. y.   m.

15 : 3) 182 : 10

By Reduction.

m.   m.

195) 2376 (12

195

———

396

396

A certain Number of Perfons, each of whofe Ages is 15 Years 3 Months (reckoning 13 Months to 1 Year) make up in all 182 Years, 10 Months; how many Perfons are there? Anfwer, 12. *viz.* The Number of times that 15 Years, 3 Months, are contained in 182 Years, 10 Months.

Queft. 3. What is the Value of 1 Yard of Cloth · · · 48 Yards coft 15 *l.* 10 *s.* 4 *d.* Anfwer, 6 *s.* , $\frac{4}{11}f.=\frac{4}{11}$ Part of 15 *l.* 10 *s.* 4 *d.* For it's plain, the Value of 1 Yard of 48 Yards, muft be the 48th Part of the Value of the whole 48 Yards, which directs us to Divifion, or taking a 48th Part of 15 *l.* 10 *s.* 4 *d.* And fo this 48, which in the mix'd Propofition is applied to Yards, is con-

		l.	s.	d.	f.
48	6	15 :	10 :	4 :	0
	8	2 :	11 :	8 :	2¼
		:	6 :	5 :	2¼¼ or ½

fidered abftractly in the Operation; which is therefore not a Divifion of Money by Yards, which cannot be made in any Senfe, but taking fuch a Part of the Money as 1 Yard is of 48 Yards, *viz.* a 48th Part.

Obferve, Had it been propofed, in the laft Queftion, to find the Value of 1 Quarter of a Yard, we may do it either by finding firft the Value of 1 Yard, and then the 4th Part of this is the Value of 1 Quarter; or, by reducing 48 Yards to Quarters, which make 192; and taking the 192d Part of the given Money. The Reafon for which is the fame as for the other Cafe.

The following Queftion *requires* all the four Operations of Arithmetick.

Queft. A Father left among 5 Sons an Eftate, confifting of 500 *l.* in Cafh; with 5 Bills, each of 48 *l.* 10 *s.* 6 *s.* He ordered 20 *l.* to be beftowed upon his Burial, and his Debts to be paid, amounting to 164 *l.* Then his free Eftate to be divided in this manner, *viz.* The eldeft Son to have the 3d Part, and the other 4 Sons to have equal Shares. What is the Share of each Son? Anfwer 186 *l.* 4 *s.* 2 *d.* to the eldeft; and 93 *l.* 2 *s.* 1 *d.* to each of the reft.

Operation.   l.

48 *l.* : 10 *s.* : 6 *d.*   20

5 Bills.   164

————————   ——

242 : 12 : 6   184

500 : 00 : 0 Cafh.

————————

742 . 12 : 6 Total.

184 : 00 : 0 deduced.

————————

3 , 558 : 12 : 6 Free Eftate.

186 : 4 : 2 Eldeft Son.

————————

4 , 372 : 8 : 4 Remains.

93 : 2 : 1 the Share of each of the other 4.

SCHOLIUM. As Queftions may be varioufly mix'd, fo the Solution will depend upon a due Confideration of the feveral Parts of the Queftion, and what Operation each may require, according to the Senfe and Effect of the different Operations in Arithmetick. But there are mix'd Applications of Multiplication and Divifion, which require other Rules and Directions, to know when and how they are to be made; thefe you'll learn in *Book* 4. and efpecially in *Book* 6. under the Name of *Proportion.* What has been already done in this Book, being fufficient for explaining the Nature of the fundamental Operations, and their more fimple Applications in whole Numbers. For the Doctrine of Fractions, you have it in the next Book.

C H A P.

C H A P. VIII.

Containing the more particular Rules of the LITERAL
ARITHMETICK, *neceſſary in the following Parts of
this Work.*

I. *For* ADDITION *or* SUBTRACTION.

CASE I. If the Numbers to be added or ſubtracted are expreſſed all by the ſame Letter, multiplied by certain Numbers, as, $3a$, or $5b$, add or ſubtract the Coefficients (*i. e.* the Numbers by which the Letters are multiplied) and multiply the ſame Letter into the Sum or Difference, it is the Sum or Difference ſought.

Examp. $3a + 5a = 8a$.   *Ex.* $2b + 3b + b = 6b$.   *Ex.* $5n - 2n = 3n$.
        Ex. $4ab + 3ab = 7ab$.   *Ex.* $3ab - ab = 2ab$.

SCHOLIUM. In order to underſtand the other Caſes, we muſt premiſe this *Obſervation,* viz. After the Addition or Subtraction of one Number to, or from another, we may ſuppoſe another added to, or ſubtracted from the preceding Sum or Difference; and another added to, or ſubtracted from the laſt Sum or Difference, and ſo on: Then is this whole Work expreſſed by ſetting the Numbers, or Letters repreſenting them, in the ſame order, with the proper Signs of the ſeveral Operations betwixt them. Thus, if b is added to a, and from the Sum c is ſubtracted, and from this Difference d ſubtracted, and to this laſt Difference e added; the Reſult of all this is expreſſed thus, $a + b - c - d + e$. But again, *Obſerve,* That if the ſame Numbers can be added or ſubtracted in any other order, the final Reſult or Effect will ſtill be the ſame; which, in all the poſſible Orders wherein the Operations can be made, is plainly the Difference betwixt the Sum of all theſe Terms that are added in the ſeveral Steps, and the Sum of all theſe that are ſubtracted: Becauſe in whatever order any Numbers are added and ſubtracted, it's evident there is ſo much in whole added, as the Sum of all theſe that are added in the ſeveral Steps, and as much ſubtracted in whole, as the Sum of all that are ſubtracted in the ſeveral Steps: Wherefore, the final Reſult is the Difference of theſe Sums. Whence, again, this follows, That 'tis no matter in what order we place the ſeveral Terms of a mix'd Expreſſion, if we always prefix the ſame Signs to the ſame Letters, and alſo take the Meaning of the Expreſſion to be univerſally the ſubtracting all theſe Terms that have — prefix'd, out of the Sum of all theſe that have + prefix'd: So that when the Operations can be performed in a propoſed Order, we may explain the Expreſſion either according to that Order, or in the preceding general way, (if that is not the propoſed Order.) And if they cannot be performed in the propoſed Order, then we explain it after the general way, as the univerſal Meaning of all ſuch Expreſſions; for tho' ſome may be explain'd another way, yet the final Reſult is always equal to this.

For *Example,* If b is greater than a, then $a - b + c$ can't be explained in the Order of theſe Letters and Signs; and if it is at all poſſible, it is ſo in this Order, $a + c - b$; yet it may repreſent the ſame thing ſtanding thus, $a - b + c$, while we do not ſo much regard

the

the Order, as the general Meaning of the Signs; which is as if it were in this Order, $a + c - b$.

C A S E II. If any complex Expreſſion [whether it is a Sum, having all its Terms joined by the Sign $+$; or a Difference, having its Terms partly $+$, partly $-$] is to be ſubtracted from another, (expreſſed ſimply or complexly) the Difference ſought may be expreſſed two ways.

Rule 1. By drawing a Line over the whole Terms of the Subtractor, and joining it to the Subtrahend with a Mark of Subtraction between them. Thus, the Difference of a and $b + c$ is expreſſed $a - \overline{b + c}$, ſignifying that b and c both, or their Sum, is taken from a, which is a quite different thing from $a - b + c$ without the Line, which would ſignify the Difference betwixt $a + c$ and b. Again, if the Subtractor is the Expreſſion of a Difference, as $b - c$ from a, the Difference ſought is expreſſed $a - \overline{b - c}$, ſignifying, that the Difference of b and c is taken from a; i. e. That c is taken from b, and the Remainder taken from a, which is a different thing from $a - b - c$, which expreſſes the Difference of a and b, c both, i. e. That b and c are both taken from a.

Now, tho' this Method is ſometimes convenient, yet it would often prove too general and indefinite, which is ſupplied by another *Rule*, whereby, from the ſimple Terms of the given Subtractor and Subtrahend, another Expreſſion is found equal to the Difference ſought. Thus,

Rule 2. Change the Signs of all the Terms of the Subtractor, and join them to the Subtrahend without a Line over them; and this expreſſes a Number equal to the Difference ſought. Thus, if the Subtractor is a Sum, as $b + c$, the Difference of this and a is $a - b - c$ $(= a - \overline{b + c}.)$

The *Reaſon* is plain; for the Sum is ſubtracted, when all the Parts are ſubtracted one after another; ſince the Parts are equal to the Whole.

Again, If the Subtractor is a Difference, as if $b - c$ is to be taken from a, the Remainder is $a - b + c$ $(= a - \overline{b - c})$ which more directly repreſents the Difference betwixt $a + c$ and b, yet is equal to the Difference of a and $b - c$.

The *Reaſon* is plain; for by adding the leſſer Term c, and then taking away the greater a, we take away as much as was before added, and alſo all that a exceeds b: Or, if b does not exceed a, we may firſt take b from a, and to the Difference add c; for thus we reſtore all that was taken away, except ſo much as b exceeds c: And ſo both ways, the Thing really taken away is preciſely what b exceeds c, or their Difference. If the Subtractor is a more complex Expreſſion, or conſiſts of more than two Terms added and ſubtracted, the Reaſon of the Rule is ſtill the ſame, from what has been explained, viz. That ſuch Expreſſions ſignify no more in Effect, than the Difference of the Sum of all that are added, and the Sum of all that are ſubtracted. Wherefore, by what's now ſhewn, all that are added in the Subtractor muſt, in expreſſing the Difference ſought, be ſubtracted; and all that are ſubtracted in it, muſt be added: Thus, $a - \overline{b + c - d - e}$ $= a - b - c + d + e$, or $a + d + e - b - c$.

S C H O L I U M. The laſt *Caſe* of this *Rule* may be conſidered as a *Theorem,* and expreſſed thus: If the Difference of two Numbers is ſubtracted from a third Number, the Remainder is the ſame, as if we added the leſſer of theſe two Numbers to the third, and from the Sum ſubtracted the greater. So $a - \overline{b - c} = a + c - b$.

C A S E III. If any complex Expreſſion of a Difference is to be added to any other Expreſſion, join them without changing their Signs, or any Line over them. *Examp.* If to a we add $b + c$, the Sum is $a + b + c$. Again, to a add $b - c$, the Sum is $a + b - c$. To $a - b$ add $c - d + e$, the Sum is $a - b + c - d + e$ $(= a + c + e - b - d.)$

The

The *Reason* of this *Rule* is evident, when the Expreffion added has all its Terms +, as in *Examp* 1. And if they are fome + fome —, as in *Ex.* 2: the Reafon is this; we are to add $b - c$; but to add b would be too much by c, therefore out of the Sum $a + b$, we muft take away c: which is the fame in Effect, as if the c had been firft taken from b, and the Difference added to a. And here *obferve*, that tho' a Line is drawn over the complex Expreffion added, it alters not the Effect: So $a + b - c$ is the fame in Effect as $a + \overline{b - c}$; fince both ways the Difference of b and c is added to a, as has been explained.

Now, to fum up thefe *Cafes* and *Rules* in one *General Rule*, it is this: To *Add*, join the Expreffions without changing their Signs; and to *Subtract*, join them, changing all the Signs of the Subtractor; or draw a Line over the Subtractor, without changing the Signs, only join the whole Expreffion thus united (and, as it were, made one Expreffion by the Line) by the Sign — betwixt it and the Subtrahend. *Laftly*, If the Numbers added or fubtracted are expreffed by the fame Letters with Coefficients, or particular Numbers prefix'd, add or fubtract thefe Numbers, and prefix the Sum or Difference to the common Literal Part.

II. For MULTIPLICATION.

CASE I. When two Numbers are expreffed by any Letters, with particular Numbers prefix'd, (or, multiplying them) then if two or more fuch are to be multiplied together, multiply all the Coefficients, and prefix the Product to the Product of the Literal Expreffions. Thus, $4a \times 3b = 12ab$; alfo, $2ab \times 5ac = 10a^2bc$.

The *Reafon* is obvious; for it is only a continual Multiplication, which may be done in any Order.

CASE II. If the Multiplier and Multiplicand, one or both, are complex Expreffions, the Product may be expreffed by the general Rule, thus: Draw a Line over the complex Terms, and join them by the general Sign of Multiplication ×. *Example*, To multiply $a + b$ by $c + d$, the Product is $\overline{a + b} \times \overline{c + d}$; or, $a + b$ by $c - d$ makes $\overline{a + b} \times \overline{c - d}$. Which would be very different Expreffions, if any of the complex Terms wanted the Line (or *Vinculum*, as the Algebraifts call it;) thus $a + b \times \overline{c + d}$ would be the Sum of a, and the Product of b into $\overline{c + d}$; and $\overline{a + b} \times c - d$, is the Difference of d, and the Product $\overline{a + b} \times c$.

So that we are to underftand the Sign of Multiplication to refer only to the firft of the fimple Terms on either hand, unlefs two or more of them are joined by a *Vinculum*. And here too *obferve*, That when feveral Letters ftand together, with, or without the Sign of Multiplication, (whereby they alfo exprefs the Product of thefe Letters) this is to be reckoned but one Term, with refpect to the following or preceding Sign, whether of Multiplication, Addition, &c. as $ab + d$, or $\overline{a + b} \times cd$. And mind alfo, that all the Terms joined together by Multiplication or Divifion, upon the Right-hand of the Sign of Addition or Subtraction, make but one Term to which that Sign refers; as $a + bc \times \overline{c + d}$ which is not to be underftood as if $a + bd$ were multiplied by $\overline{c + d}$, which then would be made $\overline{a + bc} \times \overline{c + d}$; but it is the Sum of a and the Product of bc by $\overline{c + d}$.

Obferve, Tho' this general way of reprefenting the Products of complex Expreffions is often convenient, yet there is another *Rule* more ufeful, whereby the Product is not expreffed fo indefinitely, but all reduced to more fimple Terms without any *Vinculum*.

3

Another.

Another RULE *for* Complex Expreffions.

Multiply each fimple Term of the Multiplier by each fimple Term of the Multiplicand, (according to the general Rule.) And if the Signs before each of the two fimple Terms, multiplied together, are the fame or like (*viz.* both $+$ or both $-$) prefix the Sign $+$ to the Product; but if they are unlike (*viz.* the one $+$ and the other $-$) prefix the Sign $-$. The following *Examples* fufficiently fhew the Application.

Examp. 1. $a \times \overline{b+c} = cb + ac.$ Ex. 4. $\overline{a+b} \times \overline{c-d} = ac - ad + bc - bd.$

2. $\overline{a+b} \times \overline{c+d} = ac + ad + bc + bd$ 5. $\overline{a-b} \times \overline{c-d} = ac - ad - bc + bd.$

3. $a \times \overline{b-c} = ab - ac.$

DEMONSTRATION.

1. Where all the Signs are $+$, as in *Examp.* 1. and 2. the Reafon is plain, and it has alfo been fhewn in *Lemma* 3. *Chap.* 5. *Book* 1.

2. If one of the Terms is fimple, or a Sum, and the other a Difference, as in *Ex.* 3, 4. then, for *Ex.* 3. to multiply $\overline{b-c}$ by a, it is evident that ab is too much; for we muft take only a times the Difference of b and c, or what b exceeds c; therefore if we take ac out of ab, the Remainder is a times the Number by which b exceeds c. Or thus, Let $b = c + d$, then $ab = a \times \overline{c+d} = ac + ad$ (by the 1ft *Ex.*) from which take ac, there remains ad, *viz.* the Product of a into the Difference of b and c. *Again*, If a Difference $\overline{c-d}$ is to be multiplied by a Sum $\overline{a+b}$, the Reafon is the fame for the Multiplication of each Term of the Sum into the Difference: As in *Ex.* 4.

3. If both Multiplier and Multiplicand are Differences, as in *Ex.* 5, then having multiplied $\overline{c-d}$ by a, (as in *Ex.* 3.) the Product is $ac - ad$, which is a times $\overline{c-d}$, or $\overline{c-d}$ times a: But, if inftead of $\overline{c-d}$ times a, we ought to take only $\overline{c-d}$ times $a-b$; therefore, reafoning as in *Ex.* 3. it's plain we have taken too much by $\overline{c-d}$ times b, or $cb - db$; and this being taken from the preceding Product $ac - ad$, the Remainder is the true Product, *viz.* $ac - ad - \overline{cb - db} = ac - ad - cb + db$; (by the Rules of Subtraction) which is according to the Rule.

Or we may reafon thus: Inftead of taking a times $\overline{c-d}$, which is $ac - ad$; we ought to take only $\overline{a-b}$ times $\overline{c-d}$; therefore we have taken too much by the Product of $\overline{c-d}$ by b, which is $cb - db$. Or alfo thus: Let $a = b + n$, then $a \times \overline{c-d} = \overline{b+n} \times \overline{c-d}$, $= b \times \overline{c-d} + n \times \overline{c-d}$, (by Artic. 1. taking $\overline{c-d}$ as one fingle Term.) And n being the Difference of a and b; therefore, $n \times \overline{c-d}$ is the Product fought: Confequently $b \times \overline{c-d} + n \times \overline{c-d}$, or its Equal $a \times \overline{c-d}$, exceeds it by $b \times \overline{c-d} = bc - bd$; which taken from $ac - cd$, leaves $ac - ad - bc + bd$, as before.

Obferve, If any of the Terms are more complex than thefe *Examples*, yet the Reafon is the fame in all; becaufe they are nothing elfe but a Sum or a Difference.

III. For DIVISION.

ALL the Ufe that is to be made of the Literal Divifion in the following Work, requires only, that to the General Rule I add thefe two Obfervations.

1. If the fame Letter, or Expreffion whatever, is multiplied into all the Terms of both Divifor and Dividend, by putting it out of both, the Quote is thereby expreffed more fimply. Thus, $\frac{ab}{a \times \overline{c+d}} = \frac{b}{\overline{c+d}}$ And mind, that if one Term of the Divifor or Dividend is multiplied into all the reft, then, in putting out that Multiplier, fet 1 in the Place where it ftood alone; thus, $\frac{ab+a}{ac-ad} = \frac{b+1}{c-d}$ and $\frac{3a+6}{3bc} = \frac{a+2}{bc}$, for $6 = 3 \times 2$. The Reafon of this will appear in the 4th Book. See *Chap.* 1. *General Corollary* 21. And if all the Letters reprefent Integers, fo that the Quote thus reprefented has the proper Form of a Fraction; the Reafon of this *Rule* will be feen in *Book* 2. *Chap.* 1. *Lemma* 5. *Cor.* 3.

2. If a Quote is expreffed by the Sign $\div$ or) fet betwixt Divifor and Dividend, it's to be referred only to the fimple Terms on each hand, or which have a *Vinculum*: So $a + b \div c$ is the Sum of a, and the Quote $b \div c$. But however, many fimple Terms (or fuch as become one Term by a *Vinculum*) are continuoufly joined by Signs of Multiplication and Divifion; we may explain them all in order as they ftand, one after another. Thus $a \times b \div c \times d \div e$, may be underftood thus, *viz.* a multiplied by b, and this Product divided by c, and this Quote multiplied by d, and this Product divided by e: Or thus, The Product ab divided by the Product cd, and this Quote divided by e: Or, a multiplied by the Quote $b \div c$, and this Product multiplied by the Quote $d \div e$. All which are equivalent. The Reafon of which will appear from the Rules of Fractions, when we exprefs the Quotes fractionally, thus, $a \times \frac{b}{c} \times \frac{d}{e}$; and this Method is generally moft convenient, as it leaves no Sign but that of Multiplication.

The Application of this univerfal Notation, and it's Operations, in our Reafoning about Numbers, is made by means of thefe few fimple and eafy Principles, or

A X I O M S.

1. If to any Number another be added, and from the Sum be fubtracted the Number added, the Remainder is the firft Number: Or it is the fame thing, if we firft fubtract and then add. Thus, $a + b - b = a$; or, $a - c + c = a$.

2. The fame or equal Numbers added to the fame or equal Numbers, make the Sums equal. So if we fuppofe $a = b$, then $a + d = b + d$; if $a + b = c + d$, then $a + b + n = c + d + n$; if $a = b - c$, then $a + c = b$. (For $b - c + c = b$ by the laft.)

3. The fame or equal Numbers fubtracted from the fame or equal Numbers, make the Remainders equal. So if $a = b$, then $a - d = b - d$; if $a + b = c + b$, then $a = c$; if $a + b = c$, then $a = c - b$; if $a + b = c - b$, then $a = c - 2b$.

4. If a Number is multiplied by another, and the Product divided by the fame Number, (or firft divide and then multiply) the Quote (or Product) is the firft Number. So $a \times n \div n = a$.

5. If the fame or equal Numbers are multiplied equally, the Products are equal. So if $a = b$, then $an = bn$; if $a + b - c = dn + e$, then $r \times \overline{a + b - c} = r \times \overline{dn + e}$, or $ra + rb - rc = rdn + re$; If $\frac{a}{b} = q$, then $a = bq$; (for $\frac{a}{b} \times b = a$ by the laft.) And this Cafe is the fame as the Proof of Divifion.

6. If the fame or equal Numbers are divided equally, the Quotes are equal. So if $a = b$, then $a \div d = b \div d$; or thus, $\frac{a}{d} = \frac{b}{d}$, and if $a = bq$, then $\frac{a}{b} = q$. (For $bq \div b = q$, by the 4th.)

Obferve, The Truth contained in thefe *Axioms* are univerfal, whether the fuppofed Numbers are Integers or Fractions.

BOOK

BOOK II.

Of FRACTIONS.

CHAP. I.

Containing the GENERAL PRINCIPLES *and* THEORY.

WHAT a *Fraction* is, and the *Notation* of it, has been already explained. I have shewn wherein the essential Difference betwixt *Integral* and *Fractional Numbers* does consist; and have observed, that there cannot be any other Operations in *Arithmetick* but those of *Whole Numbers*; and that the Ground and Reason of different Rules for the management of Fractions lies in their relative Nature and Value. But now more particularly *observe,* That from this relative Nature it follows, that the same Quantity may be fractionally expressed under a variety of different Forms, which in most Cases requires some preparatory Work for reducing the Numbers proposed into a like Form, before the common Operations of Addition, &c. can be performed. The first thing therefore to be done, is to explain the several Distinctions of Fractions, with the general *Theory* of their Nature; and then the Reductions of them from one Expression to another.

Observe, For brevity I contract the word *Numerator* into Num^r. and the word *Denominator* into Den^r.

DEFINITIONS.

From a Comparison of the Num^r 'to the Den^r, as a Part (taken more generally) to the Whole; Fractions are distinguished into *Proper* and *Improper.*

1. A *Proper Fraction* is that of which the Num^r is less than the Den^r, as $\frac{2}{4}$; and is called *Proper* with respect to the relative Integer, because it expresses a Quantity less than it, (as has been already explained in a *Corollary* to the Definition of a Fraction;) as if the true and proper Signification of the word Fraction were, [a Part or Quantity lesser 'than another.

2. An *Improper Fraction* is that of which the Num^r is equal to, or greater than the Den^r. as $\frac{4}{4}$, or $\frac{5}{4}$, and is called *improper* with respect to the relative Integer, because it expresses a Quantity greater than it, and is therefore not a Part of it in any sense. But taking the word *Fraction* as I have defined it, there is no such Distinction; for each Unit of the Num^r is an *Aliquot* Part of the Integer, and the Whole is a Number of such Parts: And since the applying a relative Value or Denomination to any Number, makes it a

fractional

fractional Number; therefore all such are equally true and proper, according to this larger Senfe: so that wherever this diftinction is applied, the meaning of the word *Fraction* is reftrained; or, without minding that, we may take the Terms *Proper* and *Improper*, to fignify no more than a Diftinction of thefe different Circumftances, *viz.* the *Numerator's* being lefs or not lefs than the *Denominator.*

Fractions are alfo diftinguifhed into *Simple* and *Compound.*

3. A *Simple Fraction* is one fingle Fraction, referred immediately to fome Integer; as $\frac{2}{3}$, or $\frac{3}{4}$ of any thing.

4. A *Compound Fraction* is a Fraction of a Fraction, confifting of two or more fimple Fractions referred to one another in order, and the laft referred to fome Integer; as $\frac{2}{3}$ of $\frac{4}{5}$ of any thing; or $\frac{2}{3}$ of $\frac{1}{4}$ of $\frac{5}{7}$ of any thing; the Particle, *of*, being the Mark of a Compound Fraction.

SCHOLIUM. One Fraction may be either an *Aliquot* or *Aliquant* Part of another, as well as one whole Number is of another; fo that a Fraction which is $\frac{1}{3}$ of $\frac{4}{5}$, is an *Aliquot* Part of $\frac{4}{5}$; but $\frac{2}{3}$ of $\frac{4}{5}$ is *Aliquant*.

5. A *Whole Number* with a Fraction annexed, is called a *Mix'd Number*, and if the Fraction is referred to an Unit of the fame thing that the whole Number reprefents, then, they are fet together without any mark of Addition, that being underftood; for Example $46\frac{1}{3}l.$ but if the Fraction is not referred to an Unit of the fame thing, they muft be feparated, that the name of each thing may be diftinctly apply'd, as if it were $46l. + \frac{2}{3}b.$

SCHOLIUM. In Abftract Numbers when there is no particular thing named, a *Mix'd Number* is always underftood in the firft fenfe, (*i. e.* the Fraction is fuppofed to relate to an Unit of the fame thing, which the whole Number reprefents) and fo it's written without any mark of Addition, as $24\frac{1}{2}$. Obferve alfo, that if we fuppofe (as we fhall immediately prove) that two Fractions exprefs'd in different Numbers may be equivalent, then the fame integral Number, with the fame or equivalent Fraction, makes the fame or equivalent improper Fraction.

COROLLARIES.

1. Every *Improper Fraction* is equal to fome whole or mix'd Number, and particularly, if the *Numerator* and *Denominator* are equal, the Fraction is equal to 1; for then you take as many Parts as the Integer contains, that is, the whole Integer or Unity, fo $\frac{4}{4} = 1$. And where the Num* is greater than the Denom*, the Fraction is greater than Unity, (for the Integer.) But how to reduce it, or find the equivalent whole or mix'd Number, we fhall learn afterwards.

2. Every compound Fraction is equal to fome fimple Fraction, for that which is a Part of a Part, is certainly a Part of the Whole, and we fhall fee below how to find that fimple Fraction.

AXIOMS.

1. The like Fractions of two equal Quantities or Numbers are equal; that is, if A = B then $\frac{n}{m}$ of A $= \frac{n}{m}$ of B.

2. If two Fractions are fuppofed to be equal; and if alfo one of their fimilar Terms be equal, the other is fo too; thus if $\frac{a}{b} = \frac{a}{d}$ then is $b = d$.

All the other common Axioms of Numbers, hold in Fractions as well as Integers.

L E M M A I.

The greater or leſſer the Num.^{r} of a Fraction is with the ſame Den^r, the greater or leſſer is the Value of the Fraction.　Or thus,

Two Fractions with the ſame Denom^r and different Num^{rs}, repreſent quantities of different Value; and the greateſt Num.^{r} makes the greateſt Fraction.　*Examp.* $\frac{3}{5}$ is greater than $\frac{2}{5}$ (of the ſame thing :) For the Num^r being the direct Number of things expreſſed, and the Denom^r being the ſame, the Value of each Unit of the Num.^{r} is the ſame, and therefore the greater Num^r makes the greater Value in the whole.

But more particularly, if the one Num.^{r} is Multiple of the other, that Fraction is Equi-multiple of the other; ſo $\frac{4}{9}$ is double of $\frac{2}{9}$ becauſe 4 is double of 2.　For the Value of each Unit of the Num^{rs} being equal, the compariſon of them is the ſame as if they were pure abſtract Numbers.

Coroll.　A Fraction is multiplied or divided by any Integer, if we multiply or divide its Num.^{r}, and this is a proper Multiplication or Diviſion of the direct Number of things repreſented: So $\frac{3}{7} \times 2 = \frac{6}{7}$ and $\frac{6}{7} \div 2 = \frac{3}{7}$; and univerſally $a \times \dfrac{b}{c} = \dfrac{ab}{c}$ and $\dfrac{ab}{c} \div a = \dfrac{b}{c}$ or the $\dfrac{1}{a}$ part of $\dfrac{ab}{c} = \dfrac{b}{c}$.

Obſerve, That the Diviſion is ſuppoſed here to be without a Remainder, for otherways a Fraction cannot be divided by dividing its Num.^{r}.　Becauſe the compleat Quote being a mix'd Number, it cannot be the Num.^{r} of a Fraction in proper terms.　*Obſerve* alſo that when the Num.^{r} of a Fraction is 1, it is multiplied by any Number, by placing that Number inſtead of the 1: thus, n times $\dfrac{1}{a}$ Part is $\dfrac{n}{a}$ Parts; the truth of which needs not this *Lemma*, but is comprehended in the very Nature and Idea of a Fraction.　So $\frac{3}{4}$, or 3 4th Parts, is an equivalent Expreſſion for 3 times $\frac{1}{4}$ Part, as every Number of any kind of things ſignifies ſo many Units of that kind.

L E M M A II.

Any Fraction of any Number is equal to the Sum of the like Fractions of all the leſſer Numbers of which that Whole is compoſed.　For *Examp.* $20 = 12 + 8$. therefore $\frac{1}{4}$ of 20 ($= 5$) is $= \frac{1}{4}$ of 12 ($= 3$) $+ \frac{1}{4}$ of 8 ($= 2$.)　Alſo $\frac{3}{4}$ of 20 ($= 15$) is $= \frac{3}{4}$ of 12 ($= 9$) $+ \frac{3}{4}$ of 8 ($= 6$.)　Or *Univerſally,* If $M = A + B + C$, &c. then $\dfrac{1}{m}$ of M is $= \dfrac{1}{m}$ of $A + \dfrac{1}{m}$ of $B + \dfrac{1}{m}$ of C.　Or, $\dfrac{n}{m}$ of $M = \dfrac{n}{m}$ of $A + \dfrac{n}{m}$ of $B + \dfrac{n}{m}$ of C.　[How ſeveral Fractions are added together in one ſimple Fraction, we learn afterwards; all that is deſigned here, is a general Truth, concerning a number of Fractions; for whatever way they are expreſſed, the general Idea is the ſame thing.

The *Reaſon* of this Truth is very plain; for the Whole being nothing elſe but all the Parts, when you have taken the $\frac{1}{4}$ or $\frac{1}{2}$, (or generally, $\dfrac{1}{n}$ Part, or $\dfrac{n}{m}$ Parts) of each Member of the Whole, you have taken the like Part or Parts of the Whole.

Schol.　We may alſo expreſs this Truth in this manner, *viz.* If one Quantity is made up of a number of other Quantities, $A + B + C$, and another made up of as many $a + b + c$, &c. which are reſpectively leſſer than the former, and which are each equal to the ſame Fraction of their Correſpondents in the other, (i. e. a of A, and b of B, &c.) then is the Whole $a + b + c$, &c. equal to the ſame Fraction of the Whole $A + B + C$, &c. that is, the Sum of the like Fractions of any two or more Numbers, is the like Fraction of the Sum of theſe Numbers.

COROLLARIES.

1. Any Fraction of any Number is equal to that Number of times the like Fraction of 1; for *Examp.* $\frac{1}{3}$ of 2 is $= 2 \times \frac{1}{3}$ of 1 ($= \frac{2}{3}$ of 1 *Corol. Lem.* 1.) Also $\frac{1}{7}$ of 2, is $= 2 \times \frac{1}{7}$ of 1 ($= \frac{2}{7}$ of 1.) Or *Universally,* $\frac{n}{m}$ of a is a times $\frac{n}{m}$ of 1 ($= \frac{an}{m}$ of 1, by *Cor. Lem.* 1.) For $\frac{n}{m}$ of a, is the Sum of the $\frac{n}{m}$ Parts of every lesser Number, which make up a, *i. e.* of every Unit in a, which is $\frac{n}{m}$ of 1, as often taken as there are Units in a, or a times $\frac{n}{m}$ of 1; that is, $\frac{n}{m}$ multiplied by $a = \frac{an}{m}$ of 1. (*Cor. Lem.* 1.) [And let it be always minded, that when the Fraction of a Number is proposed which has no such Fraction in pure Numbers, we must have recourse to applicate Numbers, so that the Number proposed is conceived to represent a Quantity subdivisible so as to have the Part proposed.]

SCHOL. In this *Corollary* we have a compleat Demonstration of what we have in part supposed in Division of whole applicate Numbers, and referr'd to this Place, *viz.* That any Part of any Number of Things is equal to that Number of Times the like Part of one of these Things. *Examp.* That $\frac{1}{3}$ of $2 l = \frac{2}{3}$ of $1 l$. or $\frac{1}{a}$ of $n = \frac{n}{a}$ of 1.

2. Any Fraction referred to any Number is equal to a Fraction whose Num^r is that given Number, and its Den^r the given one, and which is referred to the given Num^r as the Whole. Thus, $\frac{1}{5}$ of $2 = \frac{2}{5}$ of 3: or generally, $\frac{a}{n}$ of $m = \frac{m}{n}$ of a. For $\frac{a}{n}$ of m is $= \frac{ma}{n}$ of 1; and $\frac{m}{n}$ of $a = \frac{am}{n}$ of 1; (*per* the last.) Consequently, $\frac{a}{n}$ of m $= \frac{m}{n}$ of a.

3. From the two last this follows again, That any Fraction of any Number is an *Aliquot* Fraction having the same Den^r, and referred to the Product of the Num^r and the given Number, as the Whole: Thus, $\frac{2}{5}$ of $5 = \frac{1}{5}$ of 10. *Universally,* $\frac{a}{n}$ of $b = \frac{1}{n}$ of ab. For $\frac{a}{n}$ of $b = b$ times $\frac{a}{n}$ of 1; or, $\frac{ba}{n}$ of 1, (by the first,) and $\frac{ba}{n}$ of 1 is $= \frac{1}{n}$ of ba, (by the last;) therefore $\frac{a}{n}$ of $b = \frac{1}{n}$ of $a b$.

4. The Sum of two or more Fractions having the same Den^r, is equal to a Fraction of the same Den^r, whose Num^r is the Sum of the given Num^{rs}. *Examp.* $\frac{a}{n} + \frac{b}{n} + \frac{c}{n} = \frac{a+b+c}{n}$. For $\frac{1}{n}$ of $a + \frac{1}{n}$ of $b + \frac{1}{n}$ of $c = \frac{1}{n}$ of $\overline{a+b+c}$; but $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1. And so of the rest. Therefore, &c.

5. Here now we have a further Demonstration of that general Truth mentioned in *Schol.* after *Lem.* 2. for demonstrating Division of Whole Numbers; *viz.* That the Sum of the complete Quotes of any Numbers is equal to the complete Quote of the Sum of these Numbers, being all divided by the same Divisor. For $\frac{A+B+C}{n} = \frac{A}{n} + \frac{B}{n} + \frac{C}{n}$; and whatever these Quotes are, whether Whole Numbers or Mix'd, these Fractions express the Quotes of these Numbers by the Den^r n. But again, we see here this more general Truth, *viz.* That whatever Numbers, (Whole or Fractional) any Number (Whole or Fractional) is resolved into, the Sum of the Quotes is equal to the Quote of the Sum, being divided by the same

Integral

Integral Divisor: For whatever kind of Numbers A, B, C are, still it is true, that $\frac{1}{n}$ of A $+ \frac{1}{n}$ of B $+ \frac{1}{n}$ of C $= \frac{1}{n}$ of $\overline{A + B + C}$; and the $\frac{1}{n}$ of any Quantity is the Quote of it divided by n. Afterwards when we learn what the meaning of dividing by a Fraction is, we shall fee the fame truth hold in that Cafe alfo: So that it is Univerfal for all Cafes, whatever the Dividend and Divifor is; as you may eafily make Examples of, when you have learned the Operations of Fractions.

6. Any *Aliquot* Part of one Quantity or Number whatever, Whole or Fraction, is the fame Fraction of the like *Aliquot* Part of another, as the one Whole is of the other: Or thus, any two Numbers or Quantitys are the fame Fractions one of another as their Equimultiples. *Examp.* If $\frac{1}{3}$ of any Quantity is equal to $\frac{1}{3}$ of the $\frac{1}{5}$ of another; then the firft Whole is $\frac{1}{5}$ of the other: Or if $\frac{1}{n}$ of one Quantity is $= \frac{a}{b}$ of $\frac{1}{n}$ of another, the firft Quantity is $= \frac{a}{b}$ of the other: For each Whole being compofed of equal Parts, they are reprefented thus, $A + A + A$, &c. and $a + a + a$, &c. And the Number of Parts being equal, a and A are the like *Aliquot* Parts of their Wholes; and a is the like Fraction of A, as the Sum $a + a$, &c. is of $\overline{A + A}$, &c.

7. If we compare any Fraction, as $\frac{n}{m}$ of any Quantity, or Number whatever A, and the like Fraction of another Number B; then is $\frac{n}{m}$ of A equal to the fame Fraction of $\frac{n}{m}$ of B, as A is of B; which is plain from the laft: for $n \times \frac{1}{m}$ of A, and $n \times \frac{1}{m}$ of B, are Equimultiples of $\frac{1}{m}$ of A, and $\frac{1}{m}$ of B: And by the laft, Equimultiples, or like *Aliquot* Parts of any two Quantitys are the like Fractions one of another as thefe Quantities are; *i. e.* $n \times \frac{1}{m}$ of A ($= \frac{n}{m}$ of A) is the fame Fraction of $n \times \frac{1}{m}$ of B ($= \frac{n}{m}$ of B,) as $\frac{1}{m}$ of A is of $\frac{1}{m}$ of B: Alfo $\frac{1}{m}$ of A, is the fame Fraction of $\frac{1}{m}$ of B, as A is of B. Hence laftly, $\frac{n}{m}$ of A is the fame Fraction of $\frac{n}{m}$ of B as A of B.

L E M M A III.

The Difference betwixt the like Fractions of two Quantitys or Numbers whatever, is equal to the like Fractions of the Difference of thefe Numbers. *Examp.* $\frac{2}{3}$ of 15 ($= 10$) $- \frac{2}{3}$ of 6 ($= 4$) is $= \frac{2}{3}$ of $\overline{15 - 6}$ ($= 6$;) or generally, $\frac{n}{m}$ of A $- \frac{n}{m}$ of B $= \frac{n}{m}$ of $\overline{A - B}$. Or thus, if a, b are like Fractions of A, B, *viz.* $\frac{n}{m}$ Parts, then $a - b = \frac{n}{m}$ of $\overline{A - B}$.

DEMONST. Let A$-$B be $= d$; then is A $= d + $B: Wherefore $\frac{n}{m}$ of $\overline{A - B} = \frac{n}{m}$ of d, and $\frac{n}{m}$ of A $= \frac{n}{m}$ of $\overline{d + B}$; which is $= \frac{n}{m}$ of $d + \frac{n}{m}$ of B, (by laft *Lemma*.) Confequently, $\frac{n}{m}$ of A $= \frac{n}{m}$ of $d + \frac{n}{m}$ of B; out of each of thefe take $\frac{n}{m}$ of B; then is $\frac{n}{m}$ of A $- \frac{n}{m}$ of B $= \frac{n}{m}$ of d. But $\frac{n}{m}$ of $\overline{A - B} = \frac{n}{m}$ of d, (as above;) therefore $\frac{n}{m}$ of A $- \frac{n}{m}$ of B $= \frac{n}{m}$ of $\overline{A - B}$.

COROL.

COROL. The Difference of two Fractions of the same Den^r is equal to a Fraction of that Den^r, whose Num^r is the Difference of the given Numbers. *Example,* $\frac{a}{n} - \frac{b}{n} = \frac{a-b}{n}$, for $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1, and $\frac{1}{n}$ of $b = \frac{b}{n}$ of 1, and $\frac{1}{n}$ of $a-b = \frac{a-b}{n}$ of 1 : Therefore $\frac{a}{n} - \frac{b}{n} = \frac{a-b}{n}$.

LEMMA IV.

The more (equal) Parts any Number is divided into, the smaller these Parts are; and the fewer the Number of Parts, the greater is the Part. For *Examp.* $\frac{1}{3}$ Part is greater than $\frac{1}{4}$ Part; and so of others : For it is plain, you cannot divide the Whole into more Parts without breaking the former Parts into Pieces or smaller Parts. But more particularly, if the Den^r of one Part is Multiple of the Den^r of another, then this Part is Equimultiple of the first. For *Examp.* $\frac{1}{3}$ of any thing is double of $\frac{1}{6}$, because 6 is double of 3. And it is evident that the same reason must hold in all Cases.

Or we may express it thus; If any Quantity is divided into any Number of equal Parts, and the same (or an equal) Quantity is divided into 2, 3, 4, &c. times as many Parts, then the Part of this last Division is but $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the Part of the former. For you cannot make 2, 3, &c. times as many Parts, otherways than by breaking or dividing each of the former Parts into 2, 3, &c. whereby they will become $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Part divided; and reciprocally if the last Division is into $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the Number of Parts of the first, then is the Part of the last Division 2, 3, 4, &c. times as great as that of the first. *Universally,* $\frac{1}{a}$ Part of any thing is equal to r times $\frac{1}{ra}$ Part of that thing; and Reciprocally, $\frac{1}{ra}$ Part is but the $\frac{1}{r}$ Part of $\frac{1}{a}$ Part : So that if the Den^r of an *Aliquot* Fraction is the Product of two Numbers, that Fraction is equal to the Compound of two *Aliquot* Fractions whose Den^{rs} are these Numbers. Thus, if $a = rn$, then is $\frac{1}{a} = \frac{1}{r}$ of $\frac{1}{n}$.

LEMMA V.

If two Fractions have the same Num^r with different Den^{rs}, they represent Quantities of different Values, and that which has the greatest Den^r is the least Fraction. For *Examp.* $\frac{1}{3}$ is less than $\frac{1}{2}$, because it represents the same Number of lesser Parts, (*Lem.* 3.) But more particularly, if the greater Den^r is equal to 2, or 3, &c. times the lesser, that Fraction is but $\frac{1}{2}$ or $\frac{1}{3}$, &c. of the other. And Reciprocally, this Fraction is equal to 2 or 3 times the former; so $\frac{2}{6}$ is $\frac{1}{2}$ of $\frac{2}{3}$ (and $\frac{2}{3}$ is 2 times $\frac{2}{6}$) because 3 is $\frac{1}{2}$ of 6; and $\frac{1}{28}$ is $\frac{1}{4}$ of $\frac{1}{7}$, because 7 is $\frac{1}{4}$ of 28. *Universally,* $\frac{a}{rn}$ is $\frac{1}{r}$ of $\frac{a}{n}$, and $\frac{a}{n}$ is r times $\frac{a}{rn}$, because n is $\frac{1}{r}$ of rn.

The Reason is plain: Thus, $\frac{1}{n}$ of any thing is r times $\frac{1}{rn}$ of that thing, (*Lem.* 4.) *i. e.* $\frac{1}{n}$ of $a = r \times \frac{1}{rn}$ of a; but $\frac{1}{n}$ of $a = \frac{a}{n}$ of 1, and $\frac{1}{rn}$ of $a = \frac{a}{rn}$ of 1, (by *Cor.* 2. *Lem.* 2.) therefore $\frac{a}{n}$ of 1 = r times $\frac{a}{rn}$ of 1.

Or we may see the Truth of this somewhat otherwise; thus, $\frac{1}{n}$ and $\frac{1}{rn}$ are the like *Aliquot* Parts of $\frac{a}{n}$ and $\frac{a}{rn}$, viz. the $\frac{1}{a}$ Part; (for $\frac{1}{n} = \frac{a}{n} \div a$, or $\frac{1}{a}$ of $\frac{a}{n}$, and $\frac{1}{rn} = \frac{a}{rn} \div a = \frac{1}{a}$ of $\frac{a}{rn}$; (*Corol. Lem.* 1.) But $\frac{1}{rn} = \frac{1}{r}$ of $\frac{1}{n}$, (by *Lem.* 4.) Therefore

fore that Whole of which $\frac{1}{rn}$ is the $\frac{1}{a}$ Part, $\left(viz. \frac{a}{rn}\right)$ is alſo the $\frac{1}{r}$ Part of that Whole, of which $\frac{1}{n}$ is the $\frac{1}{a}$ Part, $\left(viz. \frac{a}{n},\right)$ *i. e.* $\frac{a}{rn}$ is the $\frac{1}{r}$ of $\frac{a}{n}$, (*Cor.* 5. *Lem.* 2) And Reciprocally, $\frac{a}{n}$ is r times $\frac{a}{rn}$.

C O R O L L A R I E S.

1. A Fraction is multiplied or divided by any Integer, if we divide or multiply its Den^r; So $\frac{a}{rn} \times r = \frac{a}{n}$, and $\frac{a}{n} \div r = \frac{a}{rn}$; For $\frac{a}{rn} = \frac{1}{r}$ of $\frac{a}{n}$, (by this *Lemma*) But the $\frac{1}{r}$ Part of any Quantity multiplied by r, produces that Quantity; therefore $\frac{a}{rn} \times r = \frac{a}{n}$; whence again $\frac{a}{n} \div r = \frac{a}{rn}$.

And take notice, That the Diviſion is ſuppoſed here to be without a Remainder; for otherwiſe the Fraction cannot be multiplied by Diviſion of its Den^r, becauſe the complete Quote is a mix'd Number, and ſo cannot be the Den^r of a Fraction.

Sch o l. As to this *Corollary*, obſerve, That by multiplying or dividing the Den^r of a Fraction, what we call properly the fractional Number is not multiplied or divided, for that is the Num^r; but the mix'd Value or Quantity expreſſed by the Fraction is multiplied or divided; ſo that it is ſtill proper to ſay, the Fraction (*i. e.* the Quantity expreſſed by it) is multiplied or divided. Again, it is manifeſtly the ſame thing in effect, to increaſe or diminiſh a Number of things, keeping the ſame Value of each; or to increaſe or diminiſh the Value of each, keeping the ſame Number; for either way the Quantity or mix'd Value of the Whole is equally increaſed or diminiſhed. Hence,

2. If the Num^r and Den^r of a Fraction are equally multiplied or divided by any Number, the Products or Quotes (where there is no Remainder) make an equal Fraction: Or thus; two Fractions are equal if the Num^r and Den^r of the one are Equimultiples, or like *Aliquot* Parts of the Num^r and Den^r of the other: So $\frac{ra}{rn} = \frac{a}{n}$. For by multiplying or dividing the Number, the Fraction is multiplied or divided in the Number of things directly expreſſed, (*Cor. Lem.* 1.) and by multiplying or dividing the Den^r by the ſame Number, the Fraction is contrarily as much divided or multiplied in the Value of the things expreſſed, (laſt *Cor.*) ſo that what the Fraction gains or loſes in the one Member, it contrarily loſes or gains as much in the other; and conſequently it remains ſtill the ſame Fraction, only in different Terms.

3. If we find a Number which will exactly divide the Num^r and Den^r of a Fraction, we can thereby reduce it to lower Terms, (*i. e.* find another Expreſſion in leſſer Numbers, which is an equal Fraction,) *viz.* by dividing the Num^r and Den^r of the given Fraction by that Number, and taking the Quotes in place of the former Num^r and Den^r. Thus, $\frac{6}{8} = \frac{3}{4}$, by dividing 6 and 8 both by 2.

Sch o l. Such Diviſors will be eaſily diſcovered in many Caſes; and from the Nature of Numbers we have theſe particular Rules for finding a Number which will divide two other Numbers, (*i. e.* the Num^r and Den^r of a Fraction) *viz.* 1. If both are even Numbers, or have in place of Units 2, 4, 6, 8, or 0; then they are both diviſible by 2; ſo $\frac{14}{28} = \frac{7}{14}$. And after one Diviſion by 2, if they are ſtill even, divide again by 2, and ſo on as long as they are even; thus, $\frac{32}{56} = \frac{16}{28} = \frac{8}{14} = \frac{4}{7}$.

2. If any one or both of them have 5 or 0 in the firſt Place, then will 5 divide them both; or if they have both o's in the firſt Places on the Right-hand, cut away an equal Num-

Number of o's from both, (which is dividing them both equally by 10 or 100, &c. according to the Number of o's cut off;) and after thefe are cut off, apply fome of the other Divifions, if the Cafe admit it.

Examp. 1. $\frac{15}{240} = \frac{3}{48}$ (*viz.* $15 \div 5 = 3$, and $240 \div 5 = 48$.)   *Ex.* 2. $\frac{20}{220} = \frac{2}{22}$.

Ex. 3. $\frac{400}{1800} = \frac{4}{18} = \frac{1}{9}$.   *Ex.* 4. $\frac{500}{1750} = \frac{50}{175} = \frac{10}{75} = \frac{2}{15}$.

When none of thefe Cafes occur, yet in fmall Numbers you will eafily difcover a Number which will divide both, if there is any fuch; and tho' 2 will divide them, yet you'll frequently find, at firft, a greater Number which will divide them.

So $\frac{7}{35} = \frac{1}{5}$, $\frac{15}{48} = \frac{5}{16}$, $\frac{24}{48} = \frac{1}{2}$, $\frac{1}{54} = \frac{1}{3}$, $\frac{24}{58} = \frac{5}{7}$.

DEFINITION 1. Two Fractions are faid to be *reverfe* or *reciprocal* to one another, when the Numr and Denr of the one is the Denr and Numr of the other, as $\frac{2}{3}$ and $\frac{3}{2}$, or generally $\frac{a}{b}$ and $\frac{b}{a}$ are Reciprocals; and becaufe any integral Number is made an improper Fraction by making 1 the Denr: Therefore a whole Number has alfo its Reciprocal, *viz.* a Fraction, whofe Denr is that whole Number, and its Numr 1. So 2 and $\frac{1}{2}$, or a and $\frac{1}{a}$ are Reciprocals.

2. Two Fractions, whereof the Terms of the one are the 2 Numrs, and the Terms of the other the two Denrs of other two Fractions, are called, The Alternate Fractions of thefe other two: Thus $\frac{A}{C}$ and $\frac{B}{D}$ are the Alternate Fractions of thefe two $\frac{A}{B}$, $\frac{C}{D}$, and thefe Alternate of thofe.

L E M M A VI.

If two Fractions are equal, then thefe Truths follow.

1. The Products made of the Numerator of each multiplied into the Denominator of the other are equal. Thus, if $\frac{a}{b} = \frac{n}{m}$ then is $am = bn$.

Examp. $\frac{2}{3} = \frac{4}{6}$, therefore, $2 \times 6 = 3 \times 4$ ($= 12$.)

DEMONST. Since $\frac{a}{b} = \frac{n}{m}$ multiply each by m, the Products muft be equal: But $\frac{a}{b} \times m = \frac{am}{b}$ (by *Cor. Lem.* 1.) and $\frac{n}{m} \times m = \frac{n}{1}$ or n, (Corol. 1. Lemma 4.) *that is,* $\frac{am}{b} = n$. Again, multiply each of thefe by b, the Products are alfo equal, *viz.* $am = bn$, which alfo follows from the Proof of Divifion.

Hence we have learned a certain Rule for trying the Equality or Inequality of two Fractions.

The *Reverfe* of this Article is alfo true, *viz.* That if the Products made of the Numerator of each Fraction, multiplied into the Denominator of the other, are equal, thefe Fractions are equal. Thus, if $am = bn$, then $\frac{a}{b} = \frac{n}{m}$. For, divide am and bn both by b, the Quotes are equal, *viz.* $\frac{am}{b} = n$; and again dividing both by m, the Quotes are equal, *viz.* $\frac{a}{b} = \frac{n}{m}$ (for $\frac{am}{b} \div m = \frac{a}{b}$.)

COROL. If $\frac{a}{b} = \frac{n}{m}$, then are n, m, both lefs or both greater, or both equal to their Correfpondents a, b; for fince $am = bn$, if n is lefs than a, b muft be greater than m, elfe it's plainly impoffible that bn fhould be $= am$; and if n is greater than a, b muft be lefs than m. *Laftly,* if $n = a$, then $b = m$.

2.

2. The *Reciprocal Fractions* are also equal; that is, if $\frac{a}{b}=\frac{n}{m}$, then is $\frac{b}{a}=\frac{m}{n}$. *Ex.* if $\frac{2}{3}=\frac{4}{6}$ then is $\frac{3}{2}=\frac{6}{4}$.

DEMONST. Because $\frac{a}{b}=\frac{n}{m}$, therefore $am=nb$, (*Art.* 1.) Divide each by a, and the Quotes are equal, *viz.* $m=\frac{bn}{a}$: then divide each of these by n, the Quotes are equal, $\frac{b}{a}=\frac{m}{n}$.

SCHOLIUM. Because the Reciprocals of two equal Fractions are equal, we may also say, that the Reciprocal of one of two equal Fractions, is a Reciprocal to the other: So if $\frac{a}{b}=\frac{n}{m}$ then are $\frac{b}{a}$ and $\frac{m}{n}$ Reciprocals, also $\frac{a}{b}$ and $\frac{m}{n}$. But then it will be convenient to distinguish betwixt immediate and remote Reciprocals. Thus $\frac{a}{b}$ and $\frac{b}{a}$ are immediate Reciprocals, and $\frac{a}{b}$, $\frac{m}{n}$, are remote Reciprocals. But when we speak of Reciprocals in general, without distinguishing, then either of the kinds may be supposed.

COROL. If the Numerator of two reciprocal Fractions are multiplied together, and also their Denominators, the Products make a Fraction equal to 1. So if $\frac{a}{b}$, $\frac{m}{n}$, are Reciprocals, then $\frac{am}{bn}=1$, for $\frac{a}{b}=\frac{n}{m}$ and $am=bn$, (Art. 1.) hence $\frac{am}{bn}=1$.

3. The *Alternate* Fractions are also equal: That is, if $\frac{a}{b}=\frac{n}{m}$, then also is $\frac{a}{n}=\frac{b}{m}$ and $\frac{n}{a}=\frac{m}{b}$. *Examp.* if $\frac{2}{3}=\frac{4}{6}$, then $\frac{2}{4}=\frac{3}{6}$.

DEMONST. Since $\frac{a}{b}=\frac{n}{m}$ then $am=bn$ (Art. 1.) divide both by n, and the Quotes are equal, *viz.* $\frac{am}{n}=b$; again, divide both these by m, and the Quotes are equal, $\frac{a}{n}=\frac{b}{m}$ (for $\frac{am}{n}\div m=\frac{a}{n}$ and $b\div m=\frac{b}{m}$:) And because $\frac{a}{n}=\frac{b}{m}$ therefore $\frac{n}{a}=\frac{m}{b}$. By *Art.* 2.

COROL. 1. If two Fractions are equivalent, as $\frac{a}{b}=\frac{n}{m}$, the Terms of the one are like Fractions of the corresponding Terms of the other: Thus n, m, are like Fractions of a, b, or a, b, like Fractions of n, m. For $n=\frac{n}{a}$ of a, and $m=\frac{m}{b}$ of b; but $\frac{n}{a}=\frac{m}{b}$: Also $a=\frac{a}{n}$ of n, and $b=\frac{b}{m}$ of m; but $\frac{a}{n}=\frac{b}{m}$.

2. If two Fractions are equivalent, as $\frac{a}{b}=\frac{n}{m}$, the Terms of the one are the Quotes of an equal Division of the correspondent Terms of the other, by some Number, either integral or fractional: For by *Cor.* 2. *Lem.* 2. $\frac{a}{n}$ or $\frac{a}{n}$ of 1 is $=\frac{1}{n}$ of a, and $\frac{b}{m}$ is $=\frac{1}{n}$ of b. Suppose then, that $\frac{a}{n}=d$, then, whether d is a whole Number or Fraction, it's plain, that n times d is $=a$, and consequently d is contained in a, n times, or $a\div d=n$. Again, $\frac{a}{n}=\frac{b}{m}$, therefore, $\frac{b}{m}=d$, and for the same Reason as before, d is contained m times in b, or $b\div d=m$. In the same manner, if we suppose $\frac{n}{a}=d$, then $n\div d=a$; and because $\frac{n}{a}=\frac{m}{b}$, therefore, $\frac{m}{b}=d$, and $m\div d=b$.

Observe,

Obferve, When we have learned afterwards, that the Quote of any Number (Whole, or Fraction,) divided by any other Number, is fuch a Fraction of the Dividend, as the Reciprocal of the Divifor expreffes, (*i.e.* the $\frac{1}{a}$ Part, if the Divifor is a whole Number a, or the $\frac{b}{a}$ Parts, if the Divifor is the Fraction $\frac{a}{b}$) then it will be plain that the one of thefe two *Corollaries* is contained in the other, fo as either of them may be deduced from the other.

L E M M A VII.

If any Number A (whole or fractional) is equal to any Fraction (proper or improper) of another Number B, then is B equal to the reciprocal Fraction of A. *Examp.* If A $= \frac{2}{3}$ of B, then is B $= \frac{3}{2}$ of A; univerfally, if A $= \frac{n}{m}$ of B, then is B $= \frac{m}{n}$ of A.

D E M O N S T. If A $= \frac{n}{m}$ of B, this fuppofes, that B being divided into m Parts, A contains n of thefe Parts; which infers reciprocally, that A being divided into n Parts, B contains m of fuch Parts.

C O R O L. Hence we have another Proof of the 2^d Article of the preceding *Lemma*, *viz.* That if two Fractions are equal, their Reciprocals are alfo equal: For if A $= \frac{n}{m}$ of B, then is B $= \frac{m}{n}$ of A; and if $\frac{n}{m} = \frac{c}{d}$, therefore A $= \frac{c}{d}$ of B, and B $= \frac{d}{c}$ of A. But fince alfo B $= \frac{m}{n}$ of A, it follows that $\frac{m}{n} = \frac{d}{c}$, elfe B would be equal to two different Fractions of A, which is impoffible.

C H A P. II.

R E D U C T I O N *of* F R A C T I O N S.

P R O B L E M I.

TO *reduce an improper Fraction to its equivalent Whole or Mix'd Number.*

Rule. Divide the Num^r by the Den^r, the Quote is the Anfwer.

   Examp. 1. $\frac{4}{4} = 1$.   *Ex.* 2. $\frac{24}{6} = 4$, (the Quote of 24 by 6.)

   Ex. 3. $\frac{14}{4} = 3\frac{2}{4}$, or $3\frac{1}{2}$, (becaufe $\frac{1}{2} = \frac{2}{4}$, *Cor.* 2. *Lem.* 4.)

D E M O N S T. The Den^r reprefents the relative Integer or Unit, expreffing it by a Number of Parts; therefore as oft as the Num^r contains the Den^r, it's equal to fo many times that Integer, (or fo many Integral Units;) and what's over in the Divifion, makes a Fraction of the given Denominator.

P R O B L E M II.

To reduce a mix'd Number, to an equivalent improper Fraction.

                Q   *Rule.*

Rule. Multiply the integral Number by the Den^r of the Fraction, and to the Product add the Num^r; make the Sum a Num^r to the given Den^r, and that is the Fraction sought.

Examp. $6\frac{2}{3}=\frac{20}{3}$; thus, $6\times 3=18$, then $18+2=20$.

DEMONST. This is plain from the laft, for it's only the Reverfe of it.

PROBLEM III.

To reduce a Whole Number to an improper Fraction, having any given Denominator.

Rule. Multiply the given Number by the given Den^r, and the Product is the Num^r of the Fraction fought.

Examp. To reduce 8 to a Fraction, having 6 for its Den^r, it is $=\frac{48}{6}$.

DEMONST. This is alfo plain from *Probl.* 2. being the Reverfe of it.

COROL. Every Whole Number is reduced to the Form of a Fraction, by making 1 the Den^r; thus, $4=\frac{4}{1}$.

SCHOLIUM. The fame or equivalent mix'd Number, *i. e.* the fame integral Number, with the fame or equivalent Fraction, will always make the fame or equivalent improper Fraction, only in different Terms, according as the fractional Part is. *And Reverfely,* The fame or equivalent improper Fraction will always reduce to the fame or equivalent mix'd Number.

Examp. $4\frac{6}{9}=\frac{42}{9}$, and $4\frac{2}{3}=\frac{14}{3}$; and becaufe $\frac{6}{9}=\frac{2}{3}$, therefore $4\frac{6}{9}=4\frac{2}{3}$, confequently, $\frac{42}{9}=\frac{14}{3}$; and from this it follows reverfely, that $4\frac{6}{9}(=\frac{42}{9})=4\frac{2}{3}(=\frac{14}{3}.)$

Hence we fee the Demonftration of a Truth propofed in *Schol.* 2. to Divifion of whole Numbers, *viz.* That the fame Quote will always be expreffed by the fame Fraction; *i. e.* That if two Numbers are propofed to be divided by other two, if the integral Quotes are the fame, when there is no Remainder; and when there is a Remainder, if the fractional Parts are alfo equivalent, then the Quotes taken fractionally (*i.e.* by fetting the Dividend as Num^r over the Divifor) will always be equal; and if the integral or mix'd Quotes are unequal, fo will thefe fractional ones be: And, in fine, whatever Part or Parts the leffer mix'd Quote is of the greater, the fame will the equivalent fractional Quotes be. So that in the Comparifon of one Quote to another, it's the fame to all Intents and Purpofes to exprefs them fractionally by the Dividends and Divifors, or to reduce (*i. e.* divide) and exprefs them directly and properly. But the Ufe and Conveniency of this way of expreffing Quotes, we fhall learn more particularly afterwards.

PROBLEM IV.

To reduce a compound Fraction to an equivalent fimple Fraction.

Rule. Multiply all the Num^{rs} continually, the laft Product is the Num^r fought; and multiply all the Den^{rs}, the laft Product is the Den^r fought.

Examp. 1. $\frac{2}{3}$ of $\frac{4}{7}=\frac{8}{21}$, (for $2\times 4=8$, and $3\times 7=21$.)

Examp. 2. $\frac{2}{5}$ of $\frac{5}{7}$ of $\frac{8}{9}=\frac{80}{315}$, (for $2\times 5\times 8=80$, and $5\times 7\times 9=315$) $=\frac{16}{63}$.

DEMONST. 1. If the compound Fraction confifts of two Parts, as $\frac{a}{n}$ of $\frac{c}{m}$; the Reafon of the Rule is this: Since $\frac{a}{n}$ Parts of any thing is $=a$ times $\frac{1}{n}$ Part, (*Cor. Lem.* 1.)

or

or alſo $= \frac{1}{n}$ Part of a times that thing, (*Cor. 2. Lem. 2.*)   Thence it's plain, that if we take firſt a times $\frac{c}{m}$, which is $= \frac{ac}{m}$, and of this take $\frac{1}{n}$ Part, which is $= \frac{ac}{nm}$; or firſt take $\frac{1}{n}$ Part of $\frac{c}{m}$, which is $= \frac{c}{nm}$; and then a times this, which is $\frac{ac}{nm}$: We have either way taken $\frac{a}{n}$ Parts of $\frac{c}{m}$, which gives the ſimple Fraction according to the Rule.

2. The ſame Reaſoning holds if the compound Fraction has three or more Members; for the two firſt being reduced to one, that one and the third make the ſame Caſe as that of two Numbers; which being reduced, gives the ſimple Fraction equivalent to the Compound of three given ones, (which will be plainly according to the Rule, *viz.* the continual Product of Numᵣˢ and Denᵣˢ) and ſo on for four or more Members.

SCHOLIUMS.

1. It's no matter whether the Members of a compound Fraction be Proper or Improper, the Reduction is done the ſame way, and for the ſame general Reaſon, wherein there is no Regard had to the Diſtinction of *Proper* and *Improper.*

But this is to be obſerved, that if all the Members are *Proper Fractions,* their equivalent ſimple Fractions will neceſſarily be *Proper;* and if they are all *Improper,* it's *Improper:* But if ſome of them are *Proper* and others *Improper,* the ſimple ones will in ſome Caſes be *Proper* and in ſome *Improper,* according as the Value of the *Proper* and *Improper* Members happen to be. But it is not to be known what it will be, otherwiſe than by applying the Rule, and actually finding the ſimple Fraction ſought. So here,

$$\tfrac{2}{3} \text{ of } \tfrac{6}{5} \text{ of } \tfrac{7}{8} = \tfrac{84}{120}; \text{ but, } \tfrac{2}{3} \text{ of } \tfrac{2}{4} \text{ of } \tfrac{7}{8} = \tfrac{126}{96}.$$

2. Fractions which are referred to a Number greater than Unity, as $\frac{2}{7}$ of 3, may be alſo conſidered as compound fractional Expreſſions (by putting the whole Number in form of a Fraction, as $\frac{2}{7}$ of $\frac{3}{1}$, reducible to a Fraction of an Unit (of the ſame things) by the ſame Rule, (and for the ſame Reaſons as before;) where it's plain we have nothing to do but multiply the Numᵣ of the Fraction by the given whole Number, and apply that Product to the given Denᵣ; ſo $\frac{2}{7}$ of $3 = \frac{6}{7}$, and $\frac{1}{4}$ of $2 = \frac{2}{4}$ ($= \frac{1}{2}$.) But the more original Reaſon for this Caſe, we have already learn'd in *Cor. 2. Lem. 2.* *Obſerve* alſo that here, as in the other kind, the ſimple Fraction will in ſome Caſes be *Proper,* and in ſome improper, even tho' the given Fraction is *Proper;* but muſt always be *Improper,* if the given Fraction is ſo. *Again,* We may have a Fraction referred to a mix'd Number, as $\frac{3}{4}$ of $5\frac{2}{7}$, and the Reduction to a ſimple Form is plainly this; Reduce the mix'd Number by *Problem* 2, and then apply the preſent *Problem,* thus, $5\frac{2}{7} = \frac{37}{7}$, and then $\frac{3}{4}$ of $\frac{37}{7}$ $= \frac{111}{28}$ ($= 3\frac{27}{28}$, *Prob.* 1.)

3. Some Authors propoſe as a kind of compound Fractions, ſuch Expreſſions wherein the Numᵣ and Denᵣ are themſelves Fractions pure or mixed; as theſe, $\dfrac{\frac{3}{4}}{\frac{2}{5}}$ or $\dfrac{9\frac{5}{7}}{13\frac{1}{2}}$ or $\dfrac{3}{5\frac{2}{7}}$. But, in my Opinion, we cannot call any of theſe a Fraction with any Propriety; for they expreſs not a certain Number of determinate Parts, which is the true and proper Notion of a Fraction. They are, indeed, reducible to an equivalent Expreſſion in the natural Form of a Fraction; but that does not make them Fractions in the proper Notion, more than a Number of Shillings can be ſaid to be an Expreſſion of *Pence,* becauſe it's reducible to ſuch an Expreſſion, (*i. e.* becauſe a Number of Pence can be aſ-

ſigned

figned equal to the given Number of Shillings.) For this Reafon I would never confider thefe Expreffions as Fractions, but only as a manner of fignifying that the one is to be divided by the other; or at moft, as an indefinite way of expreffing the Quote of the upper Number divided by the under: The finding of which Quote (or the Reduction, if you pleafe to call it fo) muft therefore be learn'd from the Divifion of Fractions. And upon the Divifion of Fractions does alfo depend another *Problem,* which fome Authors bring in among *Reductions, viz.* To find of what Number any one given Number (Whole, or Fraction, or Mix'd,) is another given Fraction. *Examp.* To find of what Number $\frac{3}{4}$ is the $\frac{2}{9}$; or to find of what Number 6 is the $\frac{3}{4}$: But thefe we muft leave to the Rule of Divifion.

4. The preceding *Rule* is general, and finds the true fimple Fraction required in all Cafes, as has been demonftrated; and that fimple Fraction may be again reduced to lower Terms, in the manner fhewn in *Cor.* 3. *Lem.* 5. But you may more eafily, in many Cafes, find the fimple Fraction required in lower Terms, at the firft, than the General Rule gives; by this Method: Firft, fee if the given fimple Fractions can be expreffed lower by the Method of *Cor.* 2. *Lem.* 5. and ufe thefe new Expreffions in place of the former, which muft certainly give the true Fraction fought; becaufe equal Fractions are the fame Fractions, only differently expreffed. *Examp.* 1. $\frac{3}{8}$ of $\frac{5}{7}$ ($=\frac{10}{42}$) is the fame as $\frac{3}{4}$ of $\frac{5}{7}$ ($=\frac{10}{21}$) becaufe $\frac{3}{8}=\frac{3}{4}$. But, 2*ly,* When you cannot reduce the given Fractions, or after you have done it, proceed thus; *viz.* Compare the feveral Num^{rs} and Den^{rs} together, and if the Num^r of one Fraction and the Den^r of another are divifible by the fame Number, (which may fometimes be the leffer of thefe two Numbers themfelves) take the Quotes and put in the Places of the Numbers divided; and do this with as many as you can; and then apply the General Rule, which will give the Fraction fought in lower Terms. The *Reafon* of which is, that by this Method you have done the fame in effect, as if you had found the fimple Fraction by the General Rule, without fuch previous Work, and then divided both Num^r and Den^r by thefe Numbers which were made Divifors in the previous Work.

The following *Examples* will illuftrate this fufficiently. I have made *Examples* only with two Members; but you can eafily do the fame when there are more Members: And as for fuch *Examples* as thefe, where there are but two Members, there will be no need to fet down the Effect of the preparatory Work, but the Anfwer of the Queftion all at once, the intermediate Steps being eafily done without writing. The finding the fimple Fraction in the fmalleft Numbers poffible, depends upon the next *Problem,* which you are to apply to the Fraction found by the preceding Rule.

Examp. 2. $\frac{4}{6}$ of $\frac{3}{7}=\frac{2}{3}$ of $\frac{3}{7}=\frac{2}{3}$ of $\frac{3}{7}=\frac{2}{7}.$ *Ex.* 3. $\frac{2}{3}$ of $\frac{3}{8}=\frac{1}{3}$ of $\frac{4}{5}=\frac{4}{15}.$

Ex. 4. $\frac{3}{7}$ of $\frac{14}{19}=\frac{3}{1}$ of $\frac{2}{19}=\frac{10}{19}.$ *Ex.* 5. $\frac{4}{12}$ of $\frac{4}{7}=\frac{1}{3}$ of $\frac{3}{7}=\frac{5}{21}.$

Ex. 6. $\frac{4}{9}$ of $\frac{6}{25}=\frac{4}{3}$ of $\frac{2}{25}=\frac{8}{75}.$ *Ex.* 7. $\frac{5}{8}$ of $\frac{20}{27}=\frac{1}{1}$ of $\frac{4}{9}=\frac{4}{9}.$

Ex. 8. $\frac{8}{15}$ of $\frac{5}{12}=\frac{2}{3}$ of $\frac{1}{3}=\frac{2}{9}.$ *Ex.* 9. $\frac{6}{7}$ of $\frac{21}{3}=\frac{2}{3}$ of $\frac{4}{1}=8.$

COROLLARIES.

1. In whatever Order the Members of a Compound Fraction are taken, it is ftill equal: So $\frac{2}{3}$ of $\frac{4}{5}=\frac{4}{5}$ of $\frac{2}{3}$; and $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{9}=\frac{7}{9}$ of $\frac{2}{3}$ of $\frac{4}{5}.$

Or alfo exchanging the Num^{rs} and Den^{rs} of any two of the Members, it is ftill equal: So $\frac{2}{3}$ of $\frac{4}{7}=\frac{2}{7}$ of $\frac{4}{3}$ In fhort, the fame Number of Simple Fractions make an equal Compound one, if the Num^{rs} of the Simples in each, and alfo the Den^{rs} are the fame Numbers, tho' in fuch Order as not to make the fame fimple Fractions. The Reafon is, becaufe

the

the Simple Fraction to which each of these Compounds is reduced, will be the same, being produced by the same Numbers.

2. Hence we learn how to diffolve a Fraction (if poffible) into two or more component Parts; *i. e.* to reduce a Simple Fraction to a Compound one: Thus, if we can difcover two, or three, or more Numbers, which multiplied together will produce a Number equal to the Num^r of the given Fraction, and as many which will produce a Number equal to the Den^r; then, of these Numbers we may make as many Simple Fractions, which, connected as the Members of one Compound Fraction, will be equal to that Simple Fraction. *Examp.* 1. $\frac{6}{20} = \frac{2}{5}$ of $\frac{3}{4}$. *Examp.* 2. $\frac{3}{16} = \frac{1}{2}$ of $\frac{3}{8}$. *Examp.* 3. $\frac{21}{24} = \frac{1}{2}$ of $\frac{21}{12}$, or $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{7}{3}$. *Examp.* 4. $\frac{3}{42} = \frac{1}{2}$ of $\frac{1}{3}$ of $\frac{3}{7}$. And this Refolution does not depend upon the Simple Fraction's being reducible to lower Terms; for this Fraction $\frac{8}{15}$, which is not reducible to lower Terms, is yet equal to $\frac{2}{3}$ of $\frac{4}{5}$. In fhort, as many Numbers as there are which will produce the Den^r, the Fraction is reducible to a Compound having as many Members, whereof thefe Numbers are the Den^{rs}; and tho' the Num^r is the Product of no Numbers but 1 and itfelf, yet that will afford as many Num^{rs} for the Members of the Compound Fraction, as in *Examp.* 2, and 4.

DEFINITION. As one of two equivalent Fractions muft be in leffer Numbers than the other, (by *Lem.* 6. *Cor.* to *Part* 1.) So that one which is expreffed by the leffer Numbers, is faid to be in *lower Terms* than the other: And a Fraction is faid to be in its *leaft* or *loweft Terms*, when there cannot be another equal to it expreffed in fmaller Numbers: So $\frac{2}{3} = \frac{4}{6}$, and $\frac{2}{3}$ is in the loweft Terms.

PROBLEM V.

To reduce a Fraction to its loweft Terms; i. e. to find an equivalent Fraction expreffed in the leaft Numbers poffible.

We have already in *Lem.* 5. *Cor.* 3. learnt how a Fraction may be reduced to lower Terms, by finding a Number (if there is any fuch) which will exactly divide both its Num^r and Denom^r: But to reduce a Fraction to its loweft Terms, (or find if it is fo already) you muft take the following

RULE. Divide the greater Term by the leffer, and the Divifor by the Remainder, and the laft Remainder by the preceding one, continually till nothing remains. Then by the laft Remainder divide the Num^r and alfo the Den^r; (in which Divifion there will be no Remainders,) the Quotes are the Terms of the Fraction fought. So that if the laft Remainder is 1, the Fraction is already in its leaft Terms.

Examp. 1.	*Examp.* 2.
$\frac{144}{560} = \frac{9}{35}$.	$\frac{7}{27}$ irreducible;

Operation.   for

```
144) 560 ( 3          7) 27 ( 3
     432                  21
    ────                  ──
    128) 144 ( 1          6) 7 ( 1
         128                 6
        ────                 ──
         16) 128 ( 8          1
             128
            ────
            000
```

Then $144 \div 16 = 9$, and
$560 \div 16 = 35$.

DEMONST. There are three things to be here demonſtrated, *viz.* 1. That the laſt Remainder will divide the Num.ʳ and Den.ʳ exactly, (or without a Remainder.) 2. That it is the greateſt Number that will do ſo. 3. That the Quotes make the leaſt equivalent Fraction.

For the firſt, I muſt premiſe theſe Truths, *viz.* 1. If any Number does meaſure (or divide without a Remainder) each of two or more Numbers, it will alſo meaſure their Sum; for it is contained in the Sum preciſely as oft as the Sum of the times it is contained in each of the Parts, (*Lem.* 2. *in Diviſion of Whole Numbers.*) Therefore the Number which meaſures another, will alſo meaſure all the Multiples of that other. 2. Every Dividend is the Sum of the Remainder, and that Multiple of the Diviſor produced by the Integral Quote, (by the Proof of *Diviſion.*) Hence, 3. If the Remainder of any Diviſion meaſure the Diviſor, it will alſo meaſure the Dividend; for it meaſures the two Parts of the Dividend, *viz.* the Remainder itſelf, and that Multiple of the Diviſor produced by the Integral Quote.

From theſe Truths we have a clear Demonſtration of the firſt thing propoſed: For in the Operation, every Diviſor and Dividend (upward from the laſt) is the Remainder and Diviſor of the laſt Diviſion: Wherefore ſince the laſt Remainder exactly divides the laſt Diviſor, it will alſo meaſure the laſt Dividend; but theſe being the Remainder and Diviſor of the preceding Diviſion, it muſt alſo meaſure the preceding Dividend; and for the ſame Reaſon, the Dividend preceding that; and ſo on it muſt meaſure every Diviſor and Dividend to the firſt, which are the Terms of the given Fraction; the thing to be proved.

For the ſecond *Article.* The laſt Remainder is the greateſt Number that will meaſure the Num.ʳ and Den.ʳ. In order to prove this, conſider, That if a Number meaſures the Sum of two Numbers, and alſo any one of them, it muſt meaſure the other; for the Sum and one Part being Multiples of that Number, ſo is the other Part, (*Corol.* 3. *Lem.* 2. *in Diviſion of Whole Numbers*) and every Number meaſures itſelf and its Multiples. But that Number which meaſures the Diviſor, meaſures any Multiple of it, *viz.* that Multiple produced by the Integral Quote, which is one Part of the Dividend; and if the ſame Number alſo meaſure the Dividend, it muſt meaſure the Remainder, which is the other Part of the Dividend. Now then if the laſt Remainder is not the greateſt Number that meaſures the Num.ʳ and Den.ʳ of the given Fraction; ſuppoſe another greater will do it: Then, by what is now ſhewn, that other will alſo meaſure the firſt Remainder, (which is the ſecond Diviſor;) and becauſe the firſt Diviſor (which this ſuppoſed Number meaſures) is the ſecond Dividend, it will alſo meaſure the ſecond Remainder; and ſo on every ſucceeding Remainder: Conſequently it will meaſure the laſt Remainder, which is abſurd; for this Number is ſuppoſed to be greater than the laſt Remainder: Wherefore the laſt Remainder is the greateſt Number which meaſures both the Numerator and Denominator.

For the laſt *Article,* viz. *That the Quotes make the Equivalent Fraction in loweſt Terms:* Let the given Fraction be expreſſed $\frac{A}{B}$, and any Fraction in lower Terms be $\frac{a}{b}$; Theſe Terms *a* and *b* are Quotes of an equal Diviſion of A, B, (by *Cor.* 2. *Lem.* 6.) But the greater the Diviſor is, the leſſer is the Quote. Therefore the greateſt Number which meaſures A and B, makes the leaſt Quotes, and conſequently the leaſt Terms of an Equivalent Fraction.

COROL. If a Fraction is not in its leaſt Terms, the Terms of it are Equimultiples of its leaſt Terms; and theſe like *Aliquot* Parts of thoſe. Hence again, All Equivalent Fractions in different Terms are different Multiples of the leaſt Terms.

PRO-

PROBLEM VI.

To reduce two or more Fractions to one Denominator; i. e. to find as many Equivalent Fractions having all the same Denominator.

R U L E. Multiply all the Den^{rs} continually into one another, the Product is the common Den^r fought. Then multiply each of the given Num^{rs} into the Den^{rs} of all the other given Fractions continually ; the Product is the Num^r of the Fraction fought, equivalent to the Fraction whofe Num^r was multiplied.

Examp. 1. $\frac{2}{3}, \frac{5}{7}, = \frac{14}{21}, \frac{15}{21}$. Thus, $3 \times 7 = 21$, the common Den^r : Then 2×7 $= 14$, which makes $\frac{14}{21} = \frac{2}{3}$. And $3 \times 5 = 15$ makes $\frac{15}{21} = \frac{5}{7}$.

Examp. 2. $\frac{5}{9}, \frac{11}{15}, \frac{8}{13} = \frac{975}{1755}, \frac{1421}{1755}, \frac{1080}{1755}$. Thus, $9 \times 15 \times 13 = 1755$, the common Den^r. $5 \times 15 \times 13 = 975$, the firft Num^r. $13 \times 9 \times 13 = 1421$, the fecond Num^r. $8 \times 15 \times 9$ $= 1080$, the third Num^r.

D E M O N S T. The Num^r and Den^r of each Fraction is equally multiplied, *viz.* by the Den^{rs} of all the other Fractions; confequently the Fractions produced are equivalent, by *Lem.* 4. *Cor.* 2.

S C H O L I U M. If it is propofed to reduce any Number of Fractions to as many equivalent Fractions in the loweft Terms that can be with a common Den^r; it is plain, that having reduced them firft according to the preceding Rule, if we can find the greateft Number that will meafure the common Den^r and all the new Num^{rs}, thefe being divided by it, the Quotes will make the Fractions fought. But the Demonftration of the Rule for finding that greateft Number muft be referred to another Place. To which I fhall therefore refer this Part of the *Problem*; and here only *obferve*, that tho' the given Fractions are in their loweft Terms, yet being reduced to a common Den^r by the prefent *Problem*, the new Fractions will not always be in their loweft Terms that admit of a common Den^r. *Examp.* $\frac{3}{5}, \frac{13}{15}$, are both in their loweft Terms; and being reduced, they are $\frac{75}{115}, \frac{117}{115}$, which are again reducible to thefe, $\frac{25}{45}, \frac{13}{45}$; for 3 meafures 75 by 25, (*i. e.* $3 \times 25 = 75$) and 117 by 39, and 135 by 45.

C O R O L. Hence we have another Demonftration of *Article* 1. *Lemma* 5. *viz.* That two Fractions are equal when the Products are equal which are made of the Num^r of each multiplied into the other's Den^r: So $\frac{2}{3} = \frac{6}{9}$, becaufe $2 \times 9 = 3 \times 6$. For when the two Fractions are reduced to one common Den^r, by the preceding Rule, thefe Products are the new Num^{rs}; and it is certain, that when two Fractions are reduced to a common Den^r, if the new Num^{rs} are alfo equal, thefe new Fractions are equal, and confequently fo are the Fractions to which they are equal; fo in the preceding Example, $\frac{2}{3}$ and $\frac{6}{9}$ being reduced, are each $= \frac{18}{27}$; therefore $\frac{2}{3}$ and $\frac{6}{9}$, which are each equal to the fame, muft alfo be equal to one another.

PROBLEM VII.

To reduce a Fraction to an Equivalent one of any other given Den^r (if poffible;) i. e. to find a Num^r which with that given Denom^r will make an Equivalent Fraction.

R U L E. Multiply the given Denom^r by the Num^r of the Fraction, and divide the Product by its Den^r; the Quote (if there is no Remainder) is the Num^r fought.

Examp. To reduce $\frac{3}{4}$ to an Equivalent Fraction, having for its Den^r 12. It is $\frac{9}{12}$. Thus, $3 \times 12 = 36$, and $36 \div 4 = 9$, the Num^r fought.

D E M O N S T. This follows from *Corol.* to the laſt. For let the given Fraction be $\frac{a}{b}$, if the Den^r to which it is to be reduced be d, ſuppoſe the Num^r fought is c. And becauſe $\frac{a}{b} = \frac{c}{d}$ by ſuppoſition, then $ad = bc$; therefore dividing both by b, it is $\frac{ad}{b} = c$, according to the Rule.

S C H O L I U M. If the Diviſion has a Remainder, the *Problem* is plainly impoſſible; yet the given Fraction is equal to the Sum of two Fractions, one of which has the given Den^r, and its Num^r is the Integral Quote of the Dividend directed, by the preceding Rule; and the other has for its Num^r the Remainder of the Diviſion, and the Den^r is the Product of the given Den^r and the Den^r of the Fraction reduced. For *Examp.* if $\frac{4}{7}$ is propoſed to be reduced to the Den^r 5, I take $4 \times 5 = 20$; then $20 \div 7 = 2$, and 6 remains. Whence I conclude, that $\frac{4}{7} = \frac{2}{5} + \frac{6}{35}$. *Univerſally,* Let it be propoſed to reduce $\frac{a}{n}$ to the Den^r m. And let $n) am = q$, and r remaining; then the *Problem* is impoſſible. But I ſay, that $\frac{a}{n} = \frac{q}{m} + \frac{r}{rm}$.

D E M O N S T. Since $\frac{am}{n} = q + \frac{r}{n}$, then dividing both by m, it is $\frac{a}{n} = \frac{q}{m} + \frac{r}{nm}$, by *Lemma* 2. For $\frac{am}{n}$ expreſſing the Sum of $q + \frac{r}{n}$ the m Part of $\frac{am}{n}$, which is the Sum, is $=$ the Sum of the m Parts of q and $\frac{r}{n}$, *i. e.* $\frac{q}{m} + \frac{r}{mn}$.

P R O B L E M VIII.

To reduce a Fraction to an Equivalent one, having a given Num^r, (if poſſible.)

R U L E. Multiply the given Num^r by the Den^r of the given Fraction, and divide the Product by its Num^r, the Quote (if there is no Remainder) is the Correſpondent Den^r fought.

Examp. To reduce $\frac{4}{6}$ to a Fraction having 18 for its Num^r; it is done thus, $18 \times 6 = 108$, and $108 \div 4 = 27$: So the Fraction fought is $\frac{18}{27}$. *Univerſally;* To reduce $\frac{a}{n}$ to the Num^r c, take $cn \div a = m$, then is $\frac{a}{n} = \frac{c}{m}$.

D E M O N S T. By reverſing the given Fraction, and taking the given Num^r as a Den^r, it becomes the ſame Caſe with the preceding *Problem*; and it has been ſhewn, that if two Fractions are equal, they are ſo when reverſed. But we may argue for this the ſame way as in that *Problem*: Thus, if $\frac{a}{n} = \frac{c}{m}$, then $am = cn$, (*Lem.* 6.) and $m = \frac{cn}{a}$.

S C H O L I U M. If there is a Remainder, the *Problem* is impoſſible; yet we can find two Fractions, the one of which has the given Num^r, and whoſe Difference is equal to the given Fraction. For which, this is the Rule; *viz.* Having multiplied the given Num^r into

into the Den^r of the Fraction, and divided the Product by its Num^r, take the Integral Quote as a Den^r to the given Num^r. And if from this Fraction you subtract another, whose Num^r is the Remainder, and the Den^r is the Product of the Den^{rs}, (*viz.* of the given Fraction and that last found) this Difference is equal to the given Fraction.

Examp. If it's proposed to reduce $\frac{4}{5}$ to a Fraction whose Num^r is 9; work thus, $7 \times 9 = 63$. Then $63 \div 5 = 12$, and 3 remains. And then I say, $\frac{4}{5} = \frac{9}{12} - \frac{3}{84}$, (84 being $= 7 \times 12$.) *Universally,* If it's proposed to reduce $\frac{a}{b}$ to the Num^r n; and if $\frac{bn}{a} = q$, with r remaining, then $\frac{a}{b}$ is not reducible to such a Num^r. But I say, $\frac{a}{b} = \frac{n}{q} - \frac{r}{bq}$.

DEMONST. Since $\frac{bn}{a} = q$, and r remaining, then is $bn = aq + r$, (by the Proof of Division.) Hence dividing equally by b, it is $n = \frac{aq + r}{b}$. And again dividing by q, it is, $\frac{n}{q} = \frac{aq + r}{bq} = \frac{aq}{bq} + \frac{r}{bq}$, (*Lem.* 2.) But $\frac{aq}{bq} = \frac{a}{b}$, (*Carol.* 2. *Lem.* 5.) Wherefore $\frac{n}{q} = \frac{a}{b} + \frac{r}{bq}$. Hence lastly, by equal Subtraction, $\frac{n}{q} - \frac{r}{bq} = \frac{a}{b}$. According to the Rule.

Observe, The preceding *Problems* relate all to *Abstract Fractions,* i. e. the Fraction reduced, and that to which it is reduced, are supposed to have the same absolute Denomination, or all to be applied to the same Integer; therefore there is none mentioned. The following *Problems* concern *Fractions* as they are specially *Applicate.*

PROBLEM IX.

To reduce a Fraction of an Unit of a higher Value, to an Equivalent Fraction of an Unit of a lower Value, these Units having a known Relation to one another, i. e. the lesser being equal to a certain known aliquot Part of the other.

RULE. Take the Reciprocal of the Fraction which expresses what Part or Parts the lower Unit is of the higher, and making that with the given Fraction (of the higher) the two Members of a Compound Fraction, reduce it to a Simple, [by *Prob.* 4.] i. e. multiply the two Num^{rs} together and the two Den^{rs}, the Products make the Fraction sought. And *observe,* if the lower is an *Aliquot* Part of the higher, we have no more to do but multiply the Num^r of the given Fraction of the higher by the Den^r of that Part.

Examp. 1. To reduce $\frac{2}{3}$ of 1 *l.* to a Fraction of 1 *sh.* it is $\frac{40}{3}$ of 1 *sh.* for 1 *sh.* is $\frac{1}{20}$ Part of 1 *l.* and 1 *l.* is 20 *sh.* Therefore $\frac{2}{3}$ of 1 *l.* is $\frac{2}{3}$ of 20 *sh.* by the Rule.

Examp. 2. To reduce $\frac{4}{5}$ of 1 *l.* to a Fraction of 1 Merk, it is $\frac{12}{10} = \frac{6}{5}$: Thus, 1 Merk is $\frac{2}{3}$ of 1 *l.* Therefore 1 *l.* is $\frac{3}{2}$ of 1 Merk, (*Lem.* 7.) So that $\frac{4}{5}$ of 1 *l.* is $= \frac{4}{5}$ of $\frac{3}{2}$ of 1 Merk, which, according to the Rule, makes $\frac{12}{10}$, the Fraction sought.

DEMONST. In the preceding Examples, I have made the Reason obvious. But to demonstrate it more *Universally*; let it be proposed to reduce $\frac{a}{b}$ of a higher Unit to a Fraction of a lower, which is $\frac{n}{m}$ of the higher. I say it is $\frac{a}{b}$ of $\frac{m}{n}$ of the lower: for since the lower is $\frac{n}{m}$ of the higher, this must be $\frac{m}{n}$ of the other, (*Lem.* 7.) Therefore $\frac{a}{b}$ of the higher is $= \frac{a}{b}$ of $\frac{m}{n}$ ($= \frac{am}{bn}$) of the lower, according to the Rule. And if

R

the

the lower Unit is an *Aliquot* Part of the higher; all we have to do, is to multiply the Num^r of the given Fraction (of the higher) by the Den^r of the *Aliquot* Part. So it is $\frac{a\,m}{z}$.

S c h o l. If a Compound Fraction, or a Fraction of a Number greater than Unity, is propofed, firft reduce it to a Simple Fraction, and then proceed as above.

P R O B L E M X.

To reduce a Fraction of a lower Unit to a higher, (the lower having a known Relation to the higher.)

R u l e. Make a Compound Fraction of the given Fraction (of the lower,) and that Fraction which expreffes what Part or Parts the lower is of the higher; and reduce this Compound to a Simple, you have the Fraction fought.

Exam. 1. To reduce $\frac{2}{3}$ of 1 *fb.* to the Fraction of 1 *l.* it is $\frac{2}{60}$ *l.* $= (\frac{1}{30}.)$ For 1 *fb.* being $\frac{1}{20}$ *l.* therefore $\frac{2}{3}$ *fb* $= \frac{2}{3}$ of $\frac{1}{20}$ of a *l.* which is $= \frac{2}{60}$ *l.* according to the Rule.

Examp. 2. $\frac{4}{5}$ of 1 Merk $= \frac{8}{15}$ of 1 *l.* For 1 Merk $= \frac{2}{3}$ *l.* therefore $\frac{4}{5}$ of 1 Merk is $= \frac{4}{5}$ of $\frac{2}{3}$ of 1 *l.* $= \frac{8}{15}$ *l.* according to the Rule.

The *Reafon* of this Rule is obvioufly the fame in all Cafes, and needs not be farther infifted on.

S c h o l. In either of the two laft *Problems,* if there are any intermediate Species betwixt the two given Units; and if inftead of the Relation betwixt the higher and lower, there be given the feveral Relations betwixt the Extremes and the Intermediate Species, then reduce the given Fraction to the firft intermediate Species, and from that to the next, till you come to the Species required. *Examp.* $\frac{2}{3}$ of 1 *l.* reduced to the Fraction of 1 Farthing, is $\frac{1920}{3}$; which is found either all at once by knowing that 1 Farthing is $\frac{1}{960}$ of 1 *l.* or by degrees thus, $\frac{2}{3} l = \frac{40}{3} fb = \frac{480}{3} d = \frac{1920}{3}$ *farthings.* By multiplying the Numerators gradually by 20, 12, and 4.

P R O B L E M XI.

To exprefs any Applicate Whole Number, fimple or mixed, by a Fraction of fome fuperiour Integer.

C a s e 1. For a Simple Number, make it the Num^r, and for Den^r take the Number of the inferiour Species which is equal to 1 of the fuperiour; and that is the Fraction fought. So 8 *d.* is $\frac{8}{12}$ of 1 *fb.* or $\frac{8}{240}$ of 1 *l.*

C a s e 2. For a mixed Number, reduce it to the loweft Species expreffed in it, and make that the Num^r; and the Number of that lower Species which is equal to 1 of the given fuperiour Species make the Den^r, and that is the Fraction fought.

Examp. To exprefs 12 *fb.* 8 *d.* 3 *f.* by the Fraction of a *l.* it is $\frac{611}{960}$ *l.* for the mixed Number is 611 *f.* and 1 *f.* is $\frac{1}{960}$ *l.* therefore 611 *f.* is 611 times $\frac{1}{960}$ *l.* $= \frac{611}{960}$ *l.*

P R O B L E M XII.

To find the Value of a Fraction of any Unit (or other Number) of a given Name, in Integers of lower Species, (where there are any fuch.)

R u l e. The given Fraction being (or made) a Simple Fraction, reduce it to a Fraction of the next lower Species, (by *Prob.* 9.) which being improper, reduce it

(by

(by *Prob.* 1.) and the Integral Quote is the Anfwer in that Species, if there is no Remainder; but if there is a Remainder, it makes a Fraction of that Species ; with which you are to proceed to the next Species, and reduce as before; and fo on to the loweft: then the Integral Number found in each Species, with the Fraction of the lower, if there is a Remainder, make up the complete Anfwer. [And *obferve*, if the Fraction of the firft, or any fucceeding lower Species is *Proper*, it is plain you can have no Integer of that Species; and fo you muft proceed, and reduce it to the next continually till you have an *Improper* Fraction : And if you never find fuch a Fraction, then the given Fraction is not expreffible in Integers.]

Examp. 1. $\frac{2}{3}$ of 1 *l.* is $= 13 \mathit{fb}$. 4 *d.* which I find thus, $\frac{2}{3} l = \frac{40}{3} \mathit{fb}$ (*Prob.* 10.) $= 13 \mathit{fb}$. $+ \frac{1}{3} \mathit{fb}$. (*Prob.* 1.) and $\frac{1}{3} \mathit{fb} = \frac{12}{3} d.$ (*Prob.* 10.) $= 4 d.$ (*Prob.* 1.) The *Reafon* of this Rule is evident of itfelf.

SCHOLIUMS.

1. This *Problem* fuppofes the given Fraction a Proper one; but for an Improper, firft reduce it, and the Integral Quote is the firft Part of the Value fought.   Then proceed with the Remainder according to the Rule.

2. This Rule is accommodated to all Cafes, whether the lower Units be *Aliquot* or *Aliquant* Parts of the higher.   But becaufe in the Cafes which moft commonly occur, they are *Aliquot* Parts, therefore the Operation is the more Simple ; and the Rule may be expreffed thus, *viz.* Reduce the Num^r of the given Fraction (as an Integer) to the next lower Species, till the Product be equal to, or greater than the Den^r; then divide by the Den^r, the Integral Quote is the Part of the Anfwer in that Species : Reduce the Remainder to the next Species, and divide as before (by the Den^r) and fo on to the loweft Species; and you have the Anfwer either in a Simple Whole Number of one Species, or Mixed of different.   And if there is a Remainder upon the laft Species, it makes that Part of the Anfwer belonging to that Species a Mixed Number with a Fraction.   And this in effect is the fame as the preceding Rule.

Examp. 2.   To find the Value of $\frac{24}{35} l.$ it is $13 \mathit{fb}$. $8 d. 2 \frac{10}{35} f.$

Operation.
$24 l.$
20

$35) 480 \mathit{fb} (13 \mathit{fb}.$
$\underline{35}$
130
105
$\overline{25} \text{ Rem.}$
12
$35) \overline{300} d (8 d.$
280
$\overline{20} \text{ Rem.}$
4
$35) \overline{80} f (2 \frac{10}{35} f.$
70
$\overline{10} \text{ Rem.}$

This way of ordering the *Operation* is diftinct and eafy ; and it is exactly according to the *General Rule* of the *Problem*, which you will readily perceive by comparing.   For this Reduction of the Remainder, and then the Divifion of the Product, is exactly the *Operation* whereby the Fraction made of that Remainder is reduced to the Fraction of the next Species, and that again reduced to a Whole Number.

3. If the Fraction to be valued refers to a Whole Number, greater than 1, as $\frac{2}{3}$ of 5 *l.* or to a Mix'd Fraction, as $\frac{2}{3}$ of $6\frac{1}{3}$ *l.* let the Expression be reduced to a simple Fraction, and then find the Value. If that simple Fraction is improper, reduce it to its equivalent Whole Number, and then find the Value of the Remainders in inferiour Species. For the valuing the Fraction of a Mix'd Whole Number, as $\frac{1}{2}$ of 48 *l.* 14 *sh.* 8 *d.* it is to be done by multiplying the Mix'd Number by the Num^r of the Fraction, and dividing the Product by the Den^r; for $\frac{1}{2}$ of any kind of Quantity is the same as $\frac{1}{4}$ of 2 times that Quantity. Or *Generally*, $\frac{a}{b}$ of any Quantity is $\frac{1}{b}$ of *a* times that Quantity, which shews the Reason of this Rule.

C H A P. III.

ADDITION *of* FRACTIONS.

DEFINITION.

ADDITION of FRACTIONS, is finding a Fraction equal to all the given Fractions taken together.

PROBLEM. *To add two or more Fractions into one Sum.*

Rule. Reduce all the given Fractions to simple Fractions, of one Unit, and one Den^r, (if they are not so already;) then the Sum of the Num^{rs} being made a Num^r to the common Den^r, makes the fractional Sum sought, (which may be further reduced as the Case admits.)

SCHOLIUM. In the following *Examples,* I thought it superfluous to write down the Operations; but I have set down the Effect of every Step in the Work, separating them from each other by the Mark of *Equality,* shewing that what follows is equivalent to what precedes; being only the same Fractions reduced (where it was necessary) to a different State, according to the Direction of the Rule: Which therefore being compared with the Rule, all will be clear and manifest.

Examp. 1. $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$.   *Ex.* 2. $\frac{2}{5}+\frac{3}{7}=\frac{14}{35}+\frac{15}{35}=\frac{29}{35}$.

Ex. 3. $\frac{2}{3}$ of $\frac{4}{5}+\frac{13}{15}=\frac{8}{15}+\frac{13}{15}=\frac{21}{15}=1\frac{6}{15}$.

Ex. 4. $\frac{3}{4}$ of $\frac{2}{7}+\frac{5}{9}$ of $\frac{1}{2}=\frac{6}{28}+\frac{5}{18}=\frac{108}{504}+\frac{140}{504}=\frac{248}{504}$.

In the preceding *Examples* the Integer is supposed to be the same in all the given Fractions, therefore I have named none; but in the following we shall make them different.

                                                  sh. d. f.

Examp. 5. $\frac{3}{7}$ *l.* $+\frac{2}{7}$ *sh.* $=\frac{60}{7}$ *sh.* $+\frac{2}{7}$ *sh.* $=\frac{62}{7}$ *sh.* $=8\frac{6}{7}$ *sh.* $=8:10:1\frac{1}{7}$.

Ex. 6. $\frac{3}{5}$ *l.* $+\frac{4}{7}$ *sh.* $=\frac{60}{5}$ *sh.* $+\frac{4}{7}$ *sh.* $=\frac{420}{35}$ *sh.* $+\frac{20}{35}$ *sh.* $=\frac{440}{35}$ *sh.* $=12\frac{20}{35}$ *sh.*

Ex. 7. $\frac{2}{3}$ *l.* $+\frac{3}{4}$ of $\frac{5}{6}$ *sh.* $=\frac{2}{3}$ *l.* $+\frac{15}{24}$ *sh.* $=\frac{60}{3}$ *sh.* $+\frac{15}{24}$ *sh.* $=\frac{1440}{72}$ *sh.* $+\frac{45}{72}$ *sh. &c.*

                                                          Again

Again, when there are mix'd Fractions, as $4\frac{2}{3} + 7\frac{1}{2}$, we may either reduce thefe to improper Fractions, and proceed by the General Rule; or, add the fractional Parts by themfelves, and the integral, and then join both their Sums. Thus in the preceding *Example*, the Fractions added make $\frac{9}{6} = 1\frac{3}{6}$, and the Integers make 11; fo the Total is $12\frac{1}{6}$. Again, take this *Example* to add $24\frac{3}{4}l$. and $16l.$ $10\frac{2}{3}fb$. The Sum of the two Fractions, *viz.* $\frac{3}{4}l$. and $\frac{2}{3}fb$. will be found $15fb.$ $4d.$ $3\frac{1}{5}f.$ The Sum of the whole Numbers is $40l.$ $10fb.$ and the Total is $41l.$ $5fb.$ $4d.$ $3\frac{1}{5}f.$ *Obferve* alfo, that if the relative Integers of two Fractions are not of one general Nature, fo as to have fome relation, there can be no Addition.

DEMONST. It is already fhewn, in *Cor.* 2. *Lem.* 2. that if feveral Fractions have one Den^r, the Sum of their Num^{rs} applied to that Den^r, is a Fraction equal to their Sum; but without that *Lemma*, this Truth will appear very fimply and eafily thus: The given Fractions being fuch, or reduced to fuch a State, that all the Num^{rs} reprefent things of the fame Denomination, both abfolute and relative, [*i. e.* of the fame Species and Value in all refpects,] their Sum is therefore a Number of things of the fame kind, or a Number of fuch Parts as the common Den^r exprefies of the fame common Integer.

CHAP. IV.

SUBTRACTION *of* FRACTIONS.

DEFINITION.

SUBTRACTION is the finding a Fraction equal to the Difference of two given Fractions.

PROBLEM. *To fubtract one Fraction from another.*

Rule. Reduce them both to fimple Fractions of one Unit, and one Den^r, (if they are not fo) then fubtract the one Num^r out of the other; and make the Remainder a Num^r to the common Den^r, and you have the fractional-Difference fought.

The *Reafon* of this *Rule* is founded upon the fame Principle as that of *Addition*, which need not be repeated. Or may alfo be deduced from *Lemma* 3. where it's fhewn that $\frac{1}{n}$ of $A - \frac{1}{n}$ of $B = \frac{1}{n}$ of $\overline{A - B}$, that is $\frac{A}{n} - \frac{B}{n} = \frac{A-B}{n}$; becaufe $\frac{1}{n}$ of $A = \frac{A}{n}$, and $\frac{1}{n}$ of $B = \frac{B}{n}$, and $\frac{1}{n}$ of $\overline{A - B} = \frac{A-B}{n}$ (*Cor.* 2. *Lem.* 2.)

Examp. 1. $\frac{3}{5} - \frac{2}{5} = \frac{1}{5}$.

Ex. 2. $\frac{3}{5} - \frac{4}{7} = \frac{11}{35} - \frac{10}{35} = \frac{1}{35}$.

Ex. 3. $\frac{3}{4} - \frac{4}{5}$ of $\frac{7}{8} = \frac{3}{4} - \frac{7}{10} = \frac{105}{140} - \frac{98}{140} = \frac{7}{140}$.

Ex. 4. $\frac{3}{4}l. - \frac{5}{6}s. = \frac{60}{4}s. - \frac{5}{6}s. = \frac{360}{24}s. - \frac{20}{24} = \frac{340}{24}s. = \&c.$

Ex. 5. $\frac{2}{3}$ of $\frac{4}{7}l. - \frac{6}{7}s. = \frac{8}{21}l. - \frac{6}{7}s. = \frac{160}{21}s. - \frac{6}{7}s. = \frac{1120}{147}s. - \frac{126}{147}s. = \frac{994}{147}s. = \&c.$

When there is a Whole Number concerned, either in the *Subtractor* or *Subtrahend*, or both, the Difference may be found alfo by the General Rule; after reducing Whole

and

and Mix'd Numbers to Improper Fractions: But such Cases may be solved easier, by reducing only the fractional Parts according to the Rule, and then subtracting Fraction from Fraction, and Whole Number from Whole Number. See *Ex.* 6, 7. below. Observing this, that where the absolute Denomination is the same, and the fractional Part of the Subtractor is greater than the Subtrahend, borrow Unity; *i. e.* add the Den^r (which represents the integral Unit) to the Num^r, and then subtract, and for that carry and add 1 to the Whole Number of the Species to which the Fraction refers: And if there is no Fraction in the Subtrahend, suppose one, whose Num^r and Den^r are equal each to the Den^r in the Subtractor, and the Fraction therefore equal to 1; and subtract from it (that is, subtract the Num^r of the Subtractor from its Den^r) and for that add 1 to the Whole Number of that Species, (*Ex.* 8, 9.) And if there is no Whole Number of that Species in the Subtrahend, or less than that to be subtracted from it, (*Ex.* 12.) you must supply it as in Subtraction of Whole Numbers. And *lastly*, mind that the Unity borrowed for the integral Part, must be repaid to the Integrals of the next Species, (*Ex.* 11.) and also the Unity which may happen to be borrowed for the Fraction of that next Species, (*Ex.* 13.) But the following *Examples* will make all clear.

Examp. 6.

Subtrahend.	$7\frac{3}{4}$	$= 7\frac{9}{12}$
Subtractor.	$4\frac{2}{3}$	$= 4\frac{8}{12}$
Differ.		$3\frac{1}{12}$

Examp. 7.

$24\frac{3}{5} = 24\frac{18}{30}$
$16\frac{5}{6} = 16\frac{25}{30}$
Diff. $4\frac{23}{30}$

Examp. 8.

14
$8\frac{3}{7}$
Diff. $5\frac{4}{7}$

Examp. 9.

46
$\frac{7}{12}$
Diff. $45\frac{5}{12}$

Examp. 10.

$22\frac{3}{4}$
16
Diff. $6\frac{3}{4}$

Examp. 11.

l.	ʃb.	d.
32 :	12 :	2
6 :	0 :	$\frac{5}{12}$
Diff. 25 :	11 :	$\frac{7}{12}$

Examp. 12.

l.	ʃb.	d.
28 :	$14\frac{6}{17}$:	9
6 :	$10\frac{3}{17}$:	$2\frac{5}{9}$
Diff. 22 :	$4\frac{3}{17}$:	$6\frac{7}{9}$

Examp. 13.

l.	ʃb.	d.		l.	ʃb.	d.
82 :	$09\frac{3}{7}$:	00	=	82 :	$09\frac{9}{21}$:	00
60 :	$14\frac{2}{3}$:	$08\frac{5}{6}$	=	60 :	$14\frac{14}{21}$:	$08\frac{5}{6}$
Diff.					$12\frac{16}{21}$:	$\frac{1}{6}$

Such *Examples* as the 12th and 13th may perhaps never (or very seldom) occur in Business, yet they are an useful Exercise to compleat one's Notion of the Nature of Fractions.

Subtraction of Fractions is proved by Addition; the same Way, and for the same Reason, as in Whole Numbers.

CHAP. V.

MULTIPLICATION *of* FRACTIONS.

DEFINITION.

TO multiply any Number or Quantity by a Fraction, is no other thing than taking fuch a Part or Parts of it as that Fraction expreffes.

PROBLEM. *To multiply one Fraction by another.*

Rule. Reduce both to fimple Fractions (if they are not fo) and then multiply the two Num.rs together, and the two Denrs; their Products make a Fraction, which is the Product fought.

Examp. 1. $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$.   *Ex.* 2. $\frac{3}{5}$ of $\frac{9}{13} \times \frac{5}{12} = \frac{135}{780} \left(= \frac{27}{156} \right)$

When either of the two Factors is a Whole Number, and the other a Fraction or Mix'd, or both being Mix'd; thefe Cafes come alfo under the preceding Rule, if the Whole or Mix'd Numbers are firft reduced to the Form of fimple Fractions, (as in the following *Examples*.) So that when there is a Whole Number to be multiplied into a Fraction, it's plain we have no more to do but multiply the Numr by that whole Number; fo 4. by $\frac{3}{5} = \frac{12}{5}$: And the Reafon of this we have alfo learn'd before, *Cor. Lem.* 1. for $\frac{3}{5}$ of 4. is 4 times $\frac{3}{5} = \frac{12}{5}$; fo that it's no matter which of thefe you call the Multiplier.

Examp. 3. $24 \times \frac{3}{5} = \frac{72}{5} \left(= 14\frac{2}{5} \right)$.

Ex. 4. $37\frac{2}{3} \times 6\frac{7}{8} = \frac{113}{3} \times \frac{55}{8} = \frac{6215}{24} \left(= 258\frac{23}{24} \right)$

Ex. 5. $64 \times 8\frac{2}{3}$ or $\frac{5}{7} = 64 \times 8\frac{10}{21} = \frac{64}{1} \times \frac{178}{21} = \frac{11104}{21} \left(567\frac{8}{21} \right)$

SCHOLIUM. After the Product is found by the General Rule, it may be reduced to lower Terms: But this Multiplication being nothing elfe than the Reduction of a compound Fraction to a fimple, we may apply the Directions in *Schol.* 4. *Probl.* 4. *Reduction*, for finding the Product in lower Terms than the General Rule gives it; of which fee the *Examples* there explained, which will be needlefs to repeat here.

DEMONST. To multiply by a Fraction fignifies no more by the Definition, than to take fuch a Part or Parts of the Multiplicand as that Fraction expreffes; *i. e.* plainly taking the given Fractions as Members of a compound Fraction, and reducing it to a Simple; which is done (fee *Probl.* 4. *Reduction*) precifely according to the Rule here given: So to multiply $\frac{2}{3}$ by $\frac{4}{5}$ is no other thing than taking $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$, (by *Probl.* 4. and this Rule.) And becaufe it's equivalent in what Order the Members of a compound Fraction are taken, therefore it's the fame which of them is called the Multiplier or Multiplicand.

For the *Proof* of this Work, it cannot have any which is a more fimple or eafier than itfelf; but it has a counter Operation in Divifion, as we fhall explain when we come to it.

For

For *APPLICATE NUMBERS.*

Any abfolute Denomination may be applied to one of the Terms; but the other muſt be abſtract by the Definition; for it can ſignify only what Part or Parts of the other are to be taken, and the Product is Applicate to the ſame Things, which is to be further reduced as the Caſe requires. For *Example*, $\frac{3}{4}l. \times \frac{5}{6} = \frac{15}{24}l. =$ 12 ſh. 6 d.

If the Multiplicand is a Mix'd, Applicate, Whole Number, and the Multiplier a Fraction, then reduce the former to the lower Species; and if there is in that Term a Fraction, make the whole an improper Fraction, and then apply the Rule.

Examp. 6. To multiply 24 l. 12 ſh. $8\frac{3}{7}d.$ by $4\frac{5}{9}$. By Reduction they are equal to

$$5912\frac{3}{7}d. \times 4\frac{5}{9} = \frac{41387}{7} \times \frac{41}{9} = \frac{1696867}{63}d. = 26934\frac{31}{63}d. = \&c.$$

But here if the Multiplier is a Whole Number, and a ſmall one, ſo that the Multiplication can eaſily be performed without previous Reduction, let it be done that way; beginning with the Fraction in the loweſt Species. *Examp.* 7. 14 l. 12 s. $10\frac{2}{5}d.$ by 4, $= 58l.$ 11 s. $5\frac{4}{5}d.$: Thus $\frac{2}{5} \times 4 = \frac{8}{5} = 1\frac{3}{5}d.$ which 1 is carried to the Product of Pence. *Examp.* 8. 78 l. 14 s. 6 d. by $\frac{5}{9}$: Multiply the Mix'd Number by 5, then divide the Product by 9; and this is a Queſtion of that ſame kind which we have ſeen already in *Schol.* 3. *Probl.* 12. *Reduction.*

GENERAL SCHOLIUM.

The word Multiplication, more properly and ſtrictly taken, ſignifies the encreaſing of a Number by Repetition; whereas to multiply by a proper Fraction (according to the preceding Rule) does plainly find a Number leſs than the Multiplicand; which does therefore rather divide than multiply it. But more particularly, when the Multiplier is an Aliquot Fraction, as $\frac{1}{3}$, the Effect of this Operation is plainly nothing elſe but Diviſion, *viz.* of the Multiplicand by 3, which finds $\frac{1}{3}$ of it. *Again*, If the Multiplier is a Fraction of any other kind, proper or improper, as $\frac{2}{3}$ or $\frac{7}{5}$, the Operation is mix'd, whereby the Multiplicand is firſt multiplied and the Product divided; for we take a certain Part of a certain Multiple of it. But this Difference is remarkable, *viz.* That if the Multiplier is a proper Fraction, the Diviſion prevails, and the Number ſaid to be multiplied is really leſſened: But if it's an Improper Fraction, the Multiplication prevails, and the Multiplicand is encreaſed. The firſt is therefore in ſome ſenſe more properly a Diviſion, and the laſt a Multiplication; tho', according to the Definition, they are both called Multiplication; (nor does the firſt agree to the Definition in Diviſion, as we ſhall ſee in the next Chapter.) *Again*, Take notice, that if a Whole Number and a Proper Fraction are multiplied together, the Fraction is, in a ſtrict and proper Senſe, multiplied; but the Whole Number is leſſened, and is only multiplied in that Senſe in which Multiplication by a Fraction is here defined. And now at laſt if you enquire, How the Name of Multiplication comes to be applied to a Work which really diminiſhes? it ſeems to be from this Conſideration, *viz.* That whether the Multiplier is a Fraction or Whole Number, the Number found has the ſame relation to the one Factor as the other has to 1; *i. e.* it contains it as oft, or as many Parts of it, as the other expreſſes, or as it contains Unity or Parts of Unity; which is plain from the Definition. So in Whole Numbers, $a \times b$ (or the Product of a by b) contains a, b times, and ſo does b contain 1, b times. In Fractions the Product of a by $\frac{n}{m}$ is $\frac{n}{m}$ Parts of a, as $\frac{n}{m}$ expreſſes $\frac{n}{m}$ Parts of 1: And becauſe of this general Likeneſs in the Effect, both are called Multiplication; which is alſo defined in this general manner, *viz.* Finding a Number which ſhall have the ſame relation (above explained) to one of the two given Numbers, as the other has to Unity: tho' the Effect of this is in ſome Caſes really Diviſion, and in all others, is mix'd of Multiplication and Diviſion, taking theſe in their more ſtrict and proper Senſe.

C H A P.

CHAP. VI.

DIVISION *of* FRACTIONS.

DEFINITION.

DIVISION is taken here in the fame Senfe as already explained in Whole Numbers, *viz.* finding how oft one Fraction is contained in another.

GENERAL SCHOLIUM.

We fee now by comparing the General Nature of Multiplication and Divifion by Fractions, as it appears in their Definitions, that they are the fame way oppofite in their Effects, as in Whole Numbers. For, let any Number A (Whole or Fraction) be multiplied by any Number B, (Whole or Fraction) and let the Product be reprefented by D; then D contains A, B times, or fuch a Fraction of A as B expreffes, (according to the Nature of Multiplication by Fractions;) confequently, if we divide D by A, *i. e.* enquire how oft A (or what Fraction of it, if it's greater than D) is contained in D, the Quote muft be B: And reverfely, if any Number (Whole or Fraction) is contained fo oft (or fuch a Fraction of it) in D, as B expreffes, then muft fo many times (or fuch a Part of) A, as B expreffes, be equal to D. Therefore, whatever be the Rule for finding how oft any Fraction is contained in another Number, this is certain, that the Quote multiplied into the Divifor (according to the Rules of Fractions) muft produce the Dividend, or its equivalent; for it may arife in different Terms, as we fhall prefently fee.

PROBLEM. *To divide one Fraction by another.*

Rule. Reduce both to fimple Fractions, then take the Dividend and the Reciprocal of the Divifor, as Members of a Compound Fraction, and multiply them together; the Simple Fraction produced is the Quote fought; *i. e.* the Quote is equal to fuch a Fraction of the Dividend as the Reciprocal of the Divifor expreffes: Or thus, (which is the fame thing) Multiply the Num^r of the Dividend into the Den^r of the Divifor, then the Den^r of the Dividend into the Num^r of the Divifor; make the firft Product the Num^r, and the other Den^r of a Fraction, and it's the Quote fought; which is to be further reduced according to the Circumftances and Senfe of the Queftion.

Examp. 1. $\frac{3}{39}$) $\frac{6}{13}$ ($\frac{234}{39}=6$.   *Ex.* 2. $\frac{2}{3}$) $\frac{8}{9}$ ($\frac{24}{18}=1\frac{6}{18}=1\frac{1}{3}$.

Ex. 3. $\frac{3}{4}$) $\frac{2}{5}$ ($\frac{8}{15}$.   *Ex.* 4. $\frac{1}{3}$ of $\frac{5}{6}$) $\frac{2}{15}$; or $\frac{5}{9}$) $\frac{8}{15}$ ($\frac{72}{75}$.

If there is a Fraction any way concerned in either of the Terms, *i. e.* if either of them is a whole Number, and the other a Fraction or mix'd, or both mix'd Numbers, they come both under the preceding Rule, if the Whole or Mix'd Number is firft reduced to the Form of a Simple Fraction, (as in the following *Examples.*) So that when either of them is a Whole Number, we have no more to do but multiply the Num^r or Den^r of the other by it, according as that whole Number is the Dividend or Divifor.

Examp.

Examp. 5. 24) $368\frac{5}{8}$ or $\frac{24}{1}$) $\frac{2113}{6}$ ($\frac{2113}{144} = 15\frac{53}{144}$.

Ex. 6. 8) $\frac{24}{39}$ ($\frac{24}{312} = \frac{1}{13}$.   *Ex.* 7. $2\frac{3}{4}$) $6\frac{5}{7}$ or $\frac{11}{4}$) $\frac{47}{7}$ ($\frac{188}{77} = 2\frac{34}{77}$.

Observe. The preceding *Examples* are all in abstract Numbers; and for the Management of applicate Numbers, where Fractions are concerned, I shall consider them by themselves, because of some things that require to be particularly explained as to the Sense and Meaning of Division applied in some Cases; but I shall first demonstrate the preceding General Rule.

DEMONST. The Reason of this Rule may be variously deduced thus : (1.) That is the true Quote, which, multiplied by the Divisor, produces the Dividend, (by what's shewn in the preceding *Gen.Schol.*) Now if $\frac{c}{d}$ is divided by $\frac{a}{b}$, the Quote according to the Rule is $\frac{bc}{ad}$, which multiplied by the Divisor $\frac{a}{b}$ (*i. e.* take $\frac{a}{b}$ of $\frac{bc}{ad}$) the Product is equal to $\frac{abc}{bad}$. But this being reduced to lower Terms, *viz.* by dividing both Num^r and Den^r by ab, (or ba,) becomes equal to $\frac{c}{d}$ the given Dividend; therefore $\frac{bc}{ad}$ is the true Quote.

Or, 2. We may prove it thus : Suppose $\frac{c}{d} \div \frac{a}{b} = \frac{n}{m}$, that is, $\frac{a}{b}$ is contained in $\frac{c}{d}$ $\frac{n}{m}$ times; wherefore $\frac{n}{m}$ times $\frac{a}{b}$, or $\frac{n}{m}$ of $\frac{a}{b} = \frac{c}{d}$: But $\frac{n}{m}$ of $\frac{a}{b} = \frac{a}{b}$ of $\frac{n}{m}$, which is therefore $= \frac{c}{d}$; and hence (by *Lem.* 7.) $\frac{n}{m} = \frac{b}{a}$ of $\frac{c}{d}$, which, according to the Rule, is the Quote of $\frac{c}{d} \div \frac{a}{b}$.

Now, tho' either of these two is a strict Demonstration of the **Rule,** yet being deduced only from the Opposition betwixt Multiplication and Division, it will be useful to see the Reason and Invention of it more directly and immediately from the Nature of Division itself. Thus,

3. Let it be required to divide $\frac{c}{d}$ by $\frac{a}{b}$. If we first enquire how oft is a contained in $\frac{c}{d}$, the Quote is $\frac{c}{da}$ (by *Cor.* 1. *Lem.* 5. *Chap.* 1.) But because we ought to enquire, how oft $\frac{a}{b}$ or $\frac{1}{b}$ of a is contained; and this must be b times as oft, therefore multiply the last Quote $\frac{c}{da}$ by b, the true Quote is $\frac{cb}{da}$ ($= \frac{b}{a}$ of $\frac{c}{d}$,) according to the Rule.

4. We have yet a more simple View of it, thus: Suppose the Divisor and Dividend have (or are reduced to) one common Den^r, then it's evident that the Dividend contains the Divisor as oft, or as many Parts of it, as its Num^r does the other; for having one Den^r, they are in the same State with respect to one another, as Whole Numbers : So that the Num^r of the Dividend, set fractionally over the Num^r of the Divisor, expresses the true Quote. For it's evident, that $\frac{a}{n}$ contains $\frac{b}{n}$ as oft as a contains b, since these Numbers express Units of the same Value. Now, tho' there is no word in the Rule of reducing to one Den^r, yet the fractional Quote, found by the Rule, is plainly the same as that now mentioned; for the Operation is the same as that by which the Num^{rs} are found, when they are reduced to a common Den^r. Thus, if $\frac{a}{b}$ and $\frac{c}{d}$ are reduced to a common Den^r,

they

they are $\frac{ad}{bd}$ and $\frac{bc}{bd}$, and the Fraction made of their Num^{rs}; *i. e.* the Number of times $\frac{ad}{bd}$ is contained in $\frac{bc}{bd}$, is truly expreffed by $\frac{bc}{ad}$, which is the Quote of $\frac{c}{d}$ divided by $\frac{a}{b}$, according to the Rule.

COROLLARIES.

If the Divifor and Dividend, being fimple Fractions, (or reduced to that State) have a common Der.^{r}, their Quote is the Quote of the Num^{rs} taken by the Rule of Divifion of whole Numbers. So $\frac{1}{7})\frac{3}{7}(\frac{3}{1}$ and $\frac{3}{7})\frac{6}{7}(\frac{6}{3}=2$:

II. From the General Rule this plainly follows: That the Reciprocal of any Quote will be the Quote when the Divifor and Dividend are changed. For *Example*, $\frac{a}{b} \div \frac{c}{d}$ $= \frac{ad}{bc}$, and $\frac{c}{d} \div \frac{a}{b} = \frac{bc}{ad}$. Or thus, if $\frac{a}{b} \div \frac{c}{d} = \frac{n}{m}$, then is $\frac{n}{m}$ of $\frac{c}{d} = \frac{a}{b}$, and $\frac{c}{d} =$ $\frac{m}{n}$ of $\frac{a}{b}$; therefore $\frac{c}{d} \div \frac{a}{b} = \frac{m}{n}$. And the fame thing being true in whole Numbers alfo, we learn this general Truth, *viz.* That if any two Numbers are divided, either by the other, the Quote of the one by the other, is the Reciprocal of the Quote of that other by the former.

SCHOLIUM. When the Quote is found, it may be reduced to lower Terms; but if you confider the Dividend and the Reciprocal of the Divifor, as Members of a compound Fraction, (according to the Rule) then you may apply the Directions given in *Schol. 4. Probl. 4. Reduction*, for finding the Quote in lower Terms. But the Rules for the more ufeful of thefe Cafes, applied to Divifion, may be expreffed in this manner, *viz.*

(1.) If the Num^r and Den^r of the Dividend are Multiples of the Num^r and Den^r of the Divifor, divide them, and make the firft Quote Num^r and the other Den^r, and that Fraction is the Quote fought.

Examp. $\frac{2}{5})\frac{4}{15}=\frac{2}{3}$.

(2.) If the Num^r of the Dividend is Multiple of the Num^r of the Divifor, but not the Den^{rs}, then take the Quote of the Num^{rs} and multiply it into the Fraction made by fetting the Den^r of the Divifor over the Den^r of the Dividend.

Examp. $\frac{2}{3})\frac{4}{7}(2\times\frac{3}{7}=\frac{6}{7}$.

(3.) If the Den^r of the Dividend is Multiple of the Den^r of the Divifor, but not the Num^{rs}, then take the Fraction made by fetting the Num^r of the Dividend over the Num^r of the Divifor, and divide it by the Quote of the Den^{rs}; *i. e.* multiply its Den^r by that Quote.

Examp. $\frac{2}{3})\frac{5}{6}(\frac{5}{2}\div2=\frac{5}{4}=1\frac{1}{4}$.

(4.) If the Num^r of the Dividend is Multiple of the Num^r of the Divifor, and the Den^r of the Divifor Multiple of the Den^r of the Dividend, divide the Num^{rs} one by the other, alfo the Den^{rs}, the Product of thofe Quotes is the Quote fought.

Examp. $\frac{3}{39})\frac{6}{13}=6$, for $6\div3=2$, and $39\div13=3$.

The *Reafon* of all thefe Rules is contained in what is explained in the place above referred to; and were fuperfluous to repeat.

For

For *A P P L I C A T E N U M B E R S.*

There is the fame Variety in Divifion of Applicate Numbers where FraƐtions are confidered, as when all are Whole Numbers. But before we make any Examples of this kind, it will be proper that we firft refleƐt upon the four different Senfes explained in Divifion of Whole Numbers, and confider how they are to be applied with FraƐtions.

Firft, If the Divifor is a Whole Number, the Dividend being a FraƐtion either pure or mixed, the different Senfes are Applicable the fame way as if the Dividend were alfo a Whole Number. For we may enquire, 1. How oft the Divifor is contained in the Dividend. 2. What Part it is of the Dividend. Or, 3. What Number (Whole or FraƐtion) is contained in the Dividend as oft as the Divifor expreffes. Or, 4. What is that Number which is fuch a Part of the Dividend as the Divifor denominates. In all which there is no matter what the Dividend is; for the Anfwer of the Queftion in the firft Senfe will anfwer it in them all; except in the fecond, when the Divifor is not an *Aliquot* Part of the Dividend; as has been already explained.

In the next Place, let the Divifor be a FraƐtion pure or mixed; whatever the Dividend is, the firft two Senfes are applicable without any Variation: for we may reafonably ask, 1. How oft a FraƐtion is contained in any Number. Or, 2. What Part it is of any Number. *Obferving* this, That whether the Divifor is an Integer or FraƐtion, if it is not an *Aliquot* Part, yet we may ask what FraƐtion it is of the other. 3. We may enquire what is that Number which is contained in the Dividend fuch a FraƐtion of a time (*i. e.* of which the Dividend contains, or is equal to fuch a FraƐtion) as the Divifor expreffes. It is plain, the Quote taken in the firft Senfe will anfwer this alfo; becaufe the Quote and Divifor produce the Dividend. For let $\frac{c}{d} \div \frac{a}{b}$ be $= \frac{n}{m}$, which in the firft Senfe fignifies that $\frac{a}{b}$ is contained in $\frac{c}{d}$, $\frac{n}{m}$ Parts of a time; *i. e.* that $\frac{c}{d} = \frac{n}{m}$ of $\frac{a}{b}$. But $\frac{n}{m}$ of $\frac{a}{b} = \frac{a}{b}$ of $\frac{n}{m}$, (by *Cor.* 1. *Probl.* 4. of ReduƐtion.) Therefore $\frac{c}{d} = \frac{a}{b}$ of $\frac{n}{m}$, *i. e.* $\frac{c}{d}$ contains $\frac{n}{m}$, $\frac{a}{b}$ Parts of a time. Wherefore $\frac{n}{m}$ is the Quote in this third Senfe alfo. And this Queftion applied to a fraƐtional Divifor, is parallel to the third Senfe applied to an Integral Divifor. And to comprehend both without diftinguifhing, we may ask what is the Number which is contained in the Dividend fo many times, or fuch a FraƐtion of a time as the Divifor expreffes.

4. The fourth Senfe in Whole Numbers is finding a Number which is fuch a Part of the Dividend as the Divifor denominates; which is in effeƐt multiplying by its reciprocal FraƐtion; for it's plainly taking fuch a FraƐtion of the Dividend as the Reciprocal of the Divifor expreffes. So to divide by 4, in this Senfe, is to take $\frac{1}{4}$. And to make a Queftion like this with a fraƐtional Divifor, we muft feek a Number which is fuch a FraƐtion of the Dividend as the Reciprocal of the Divifor expreffes; which is the immediate effeƐt of the Rule given for finding the Quote in the firft Senfe: So that it is plain, the Quote which anfwers the Queftion in the firft Senfe, does fo in this Senfe alfo; *i. e.* is a Number which is fuch a FraƐtion of the Dividend as the Reciprocal of the Divifor expreffes. This therefore is a general Truth, that whatever the Divifor and Dividend be, the Quote found by the general Rule, is fuch a FraƐtion of the Dividend as the Reciprocal of the Divifor expreffes.

But laftly *obferve*, That if a Queftion is thus propofed, *viz.* to find $\frac{b}{a}$ Parts of $\frac{c}{d}$, it is direƐtly in the Form of Multiplication, and fo does not appear as a Queftion of Divifion; nor is it fo in any other Senfe than as the Anfwer of it is equal to the Number of

3

times

times that its Reciprocal $\frac{a}{b}$ is contained in $\frac{c}{d}$: Or, as it is the Anfwer of a Queftion of Divifion made with the Divifor $\frac{a}{b}$ in the third Senfe.

Now then, as in Divifion of Applicate Whole Numbers, fo here in Fractions there are but two Cafes : For either,

1. The Divifor and Dividend are both Applicate to the fame kind of thing; and the Queftion is always in the firft and fecond Senfe: Therefore the Quote is an Abftract Number.   And to find it, you muft reduce the given Numbers to one abfolute Denomination or Integer, and then apply the preceding general Rule.   See *Examp.* 8, 9, 10. *below.*

2. The Dividend is Applicate, and the Divifor Abftract; then the Queftion can be propofed only in the third and fourth Senfe.   Minding that the fourth Senfe is to be called Divifion only as it is the Anfwer of the Queftion propofed in the third Senfe, with the Reciprocal of the Divifor,   See *Examp.* 14, 15, 16. *below.*

Examples of CASE I.

Ex. 8. $\frac{2}{3}l.$) $\frac{8}{5}l.$ ($\frac{4}{3}=1\frac{1}{3}.$ *Ex.* 10. $3\frac{5}{9}l.$) $18\frac{3}{7}fb.$ Or, $\frac{32}{9}l.$) $\frac{129}{7}fb.$

Ex. 9. $\frac{3}{5}fb.$) $\frac{2}{3}l.$ Or, $\frac{640}{9}fb.$) $\frac{129}{7}fb.$ ($\frac{1161}{4480}.$

By Reduction.   *Ex.* 11. $3\frac{2}{5}fb.$) $26l.$ Or, $\frac{17}{5}fb.$) $\frac{520}{1}fb.$ ($\frac{2600}{17}=152\frac{16}{17}.$

$\frac{3}{5}fb$) $\frac{40}{3}fb$ ($\frac{200}{9}=22\frac{2}{9}.$

If either Divifor or Dividend is a mix'd Whole Number of different Species, reduce both to a Simple Number of the loweft Species; and if there is a Fraction in one or both, they muft alfo be reduced to one Integer, and that the fame to which the Integral Numbers are reduced.   But this will be beft done by reducing to an Improper Fraction, and then reducing this to the Integer required.

Ex. 12. $4\frac{3}{5}fb.$) $48l.$ $18s.$ $9d.$ By Reduction, $\frac{276}{5}d$) $\frac{11745}{1}d$ ($\frac{58725}{276}=$ $212\frac{214}{276}.$

Ex. 13. $3l.$ $4\frac{2}{7}fb.$) $38l.$ $12fb.$ $8\frac{1}{5}d.$
                By Reduction.
$64\frac{2}{7}fb.$) $9272\frac{1}{5}d.$ Or, $\frac{450}{7}fb.$) $\frac{46361}{5}d.$ Or, $\frac{450}{7}fb.$) $\frac{46361}{60}fb.$ ($\frac{324527}{27000}$
$=12\frac{527}{27000}.$

Examples of CASE II.

Where the Dividend is Applicate, and the Divifor Abftract; i. e. wherein is fought a Number, of which fuch a Fraction or Multiple as the Divifor expreffes, is equal to the Dividend.

Examp. 14. $\frac{2}{3}$) $\frac{8}{5}l.$ ($\frac{4}{3}l.=1\frac{1}{3}l.=1l.$ $6fb.$ $8d.$

Examp. 15. $4\frac{3}{5}$) $24l.$ $16s.$ $9d.$ Or, $\frac{23}{5}$) $\frac{5961}{1}d.$ ($\frac{29805}{23}d.=1295\frac{20}{23}d.=$
$5l.$ $7s.$ $11\frac{20}{23}d.$

If.

If the Divifor is a Whole Number, you need not always reduce the Dividend when it is a mixed Number, but carry on the Divifion by degrees to the loweft Species, where the Fraction is; and what is Remainder upon dividing the Integral Part of that Species, annex to the Fraction, and reduce them to an Improper Fraction, and then divide as is done in the following Example.

Examp. 16.   4) 18 *l.* 13 *fb.* $9\frac{2}{3}$ *d.* (4*l.* 13*fb.* $5\frac{5}{12}$ *d.*

GENERAL SCHOLIUM.

As we obferved before upon the multiplying by a proper Fraction, that it diminifhes the Number multiplied by it, contrary to the effect of an Integral Multiplier, and to the more limited Senfe of the word Multiplication; fo here, to divide by a proper Fraction does always quote a Number greater than the Dividend, contrary to what is done with an Integral Divifor; and contrary alfo to the more ftrict Senfe of the word Divifion, which imports the leffening of a thing: and fo the Quote will in fome Cafes be a Whole or Mix'd Number, tho' the Dividend is a Fraction; becaufe one Fraction may be equal to another taken once, twice, or any Number of Times, or Parts over. Now, that the Quote muft be always greater than the Dividend, is plain from the Rule. For the Divifor being a Proper Fraction, its Reciprocal is an Improper Fraction greater than Unity, by which the Dividend is multiplied, which is therefore taken more than once to make the Quote. Again more particularly, if the Divifor is an *Aliquot Fraction*, the Operation is in effect a proper Multiplication; fo to divide by $\frac{1}{4}$, is no other thing than multiplying by 4. And the Divifor being a Fraction of any other kind, the Operation is mix'd of a proper *Multiplication* and *Divifion*, in which the Multiplication prevails, if the Divifor is a proper Fraction; but the Divifion prevails, if the Divifor is an improper Fraction. Now, becaufe the Quote may be in fome Cafes greater than the Dividend, we may enquire, How the name *Divifion* came to be applied to an Operation which really increafes a Number: And the Reafon of this is probably, (as before we alledged in Multiplication,) the general likenefs in the effect of dividing by a Whole Number and by a Fraction, *viz.* That the Quote has the fame Relation to the Dividend that Unity has to the Divifor, *i. e.* that the Quote contains the fame Part or Parts of the Dividend as Unity does of the Divifor. This is manifeft in Whole Numbers; for if B is divided by A, the Quote is $\frac{1}{A}$ of B, and 1 is $\frac{1}{A}$ of A. In Fractions the fame truth will exactly appear; Thus, let the Dividend be D, (either a Whole Number, or Fraction,) and the Divifor $\frac{a}{b}$; by the Rule, the Quote is $= \frac{b}{a}$ of D; but 1 is alfo $= \frac{b}{a}$ of $\frac{a}{b}$; (thefe being Reciprocals, *Cor. Lem.* 5.) And from hence fome define *Divifion* in this manner, viz. *Finding a Number which fhall have the fame Relation* (as before explained) *to the Dividend, as Unity has to the Divifor.*—— But we have alfo this Difference to obferve, in the Definitions which I have applied to *Multiplication* and *Divifion* : That in *Divifion*, the Definition is the fame for Fractions and Whole Numbers without changing one Word, *viz.* finding how oft the *Divifor* is contained in the *Dividend.* But in *Multiplication* there is Variety; for in Whole Numbers it is repeating the Multiplicand, and taking the Whole of it a Number of Times; but with a Fraction it is only taking a Part, or certain Parts of it. Yet this is ftill true, That the Operations of *Multiplication* and *Divifion* are reverfe to one another; fo that the one undoes what the other did, whereby they prove one another. And hence we have this

G E-

GENERAL COROLLARY, *To the Rules of Multiplication and Divifion, both by Whole Numbers and Fractions;* viz.

To multiply by any Number, Whole, or Fraction, and to divide by its Reciprocal, have always the fame effect, or brings out the fame Number; fo $8 \times 3 = 8 \div \frac{1}{3}$; or generally, $A \times m = A \div \frac{1}{m}$, each being (by their proper Rules) $= A m$; or, $\frac{A}{B} \times m = \frac{A}{B} \div \frac{1}{m}$, each being $= \frac{A m}{B}$. And $A \times \frac{n}{m} = A \div \frac{m}{n}$; for each is $= \frac{A n}{m}$. Alfo $\frac{A}{B} \times \frac{n}{m} = \frac{A}{B} \div \frac{m}{n}$, each being $\frac{A n}{B m}$.

The laft thing I have to *obferve*, is, That fome may ask, Why the finding, for Example, $\frac{1}{3}$ of a thing is not called *Divifion*, fince $\frac{1}{3}$ is found by dividing, (*viz.* with 3.) The Anfwer is plainly this, Dividing by 3 is the fame as multiplying by $\frac{1}{3}$. So that to find $\frac{1}{3}$ we do multiply according to the Rules of Fractions, which makes the Product a Fraction; and when this happens to be an Improper Fraction, (as it is always when the Dividend is a Whole Number greater than 3) the Divifion by 3 is only the Reduction of that Improper Fraction. And for the fame Reafon to find $\frac{2}{3}$ will alfo require a Divifion, if the Fraction produced be Improper: But to make the Comparifon aright, let them confider the Difference betwixt dividing by 3 and by $\frac{1}{3}$; thefe cannot have the fame Effect. And if dividing by 3, finds $\frac{1}{3}$ of the Dividend; dividing by $\frac{1}{3}$ does not alfo find $\frac{1}{3}$ of it. Indeed, the finding $\frac{1}{3}$ is in fome fenfe *Divifion*; but it is not Divifion by $\frac{1}{3}$, but by 3. Or it is not finding how oft $\frac{1}{3}$, but how oft 3 is contained in the Dividend. In the fame manner, finding $\frac{2}{3}$ is in a fenfe *Divifion*, but it is not Divifion by $\frac{2}{3}$, but by $\frac{3}{2}$ its Reciprocal; in the fame manner as $\frac{1}{3}$ is found by dividing by 3, the Reciprocal of $\frac{1}{3}$. In fhort, we muft take *Multiplication* and *Divifion* according to their eftablifhed Definitions; about which I have faid all that is neceffary to make them be clearly underftood.

CHAP. VII.

Of the more fpecial Application of Fractions.

I Have in the preceding Chapter not only explained the general Principles and Operations in Fractions, but alfo made Application to particular things by Examples in all the Operations. Thefe Examples are indeed Simple, and but pure Suppofitions: And if we confider that Fractions require more Operation than abfolute Numbers, it is unreafonable to bring them into Bufinefs without neceffity. It remains then that I make a few further Reflections, and fhew you how they neceffarily occur in Bufinefs. In the firft place obferve, that in what we call mix'd Whole Numbers, the inferiour Species are indeed Fractions, but fuch as we may call *tacite ones*; becaufe their Denominations are never exprefs'd, it being thought more convenient to diftinguifh them by common Names than by numeral Denominations; yet thefe are always underftood, and really applied in all Operations: And the Rules given about them in the *firft Book*, are either the very fame, or deduced from the general Rules of Fractions explained in this *Book*; and fo have the fame effect: For fince each of thefe Rules is demonftrated to be true and

3 right,

right, the Effect of each muſt be the ſame, when applied to the ſame things. But I ſhall alſo ſhew it by a ſhort Compariſon, thus :

In *Addition* of mix'd Whole Numbers, we add the Numbers of each Species by themſelves without any further preparation, becauſe they are Fractions of the ſame Integer, *viz.* the next higher Species, having all one common Denominator, (or numeral Relation to the next Species,) which is exactly the Rule of Addition of Fractions; and then if the Sum is greater than that Der.^{r}, it's an Improper Fraction; and accordingly we do in effect reduce it to an Integral Number of that Species, in the very Addition, (which is equivalent to dividing it by the common Der.^{r}) leaving the Remainder in the loweſt Species to which it belongs. So Pence are 12th Parts of a Shilling, and therefore we carry the Number of 12's contained in the Sum of Pence to the Shillings ; and ſo on.

In *Subtraction* we do the ſame in effect as the Rule for Fractions requires; for the like Species are Fractions having the ſame Der.^{r}, (with reſpect to the next above,) and when the Number in the Subtrahend is leaſt, we borrow Unity from the next Species, *i. e.* the Den^r, (as in *Examp.* 7, 8, 9. *Subtraction of Fractions.*)

In *Multiplication*, which is but a reiterated Addition, the Compariſon is the ſame as in Addition, if the Work is performed by beginning at the lower Species, and multiplying upwards; and if we reduce firſt the Mix'd Number to a Simple Number of the loweſt Species, then it is a Fraction of the higheſt, whoſe Den^r is the Relation betwixt the loweſt Species and higheſt. And this Fraction is multiplied by multiplying its Num^r, (*i. e.* the Number produced by the Reduction,) and the Product makes an Improper Fraction; the Reduction of it being nothing elſe but finding its equivalent Mix'd Number, and again reducing the Integral Part to the higher Species.

In *Division*, if the Dividend is a Mix'd Whole Number, and the Diviſor a ſimple Abſtract Number, the Compariſon is the ſame as in *Multiplication*, if we take the Method of reducing the mix'd Number to the loweſt Species; and if we divide from the higheſt Species gradually, then it is the ſame as if we expreſs'd each Species as a Fraction of the next above it, and divided each Member by itſelf. Again, if the Diviſor and Dividend are both Applicate, whether both are mix'd, or only one, the Reduction of both to the ſame loweſt Species, is making them both Fractions of the higheſt Species and of one Den^r, (*viz.* the Relation betwixt the higheſt and loweſt,) and then dividing their Num^{rs}, which is according to the Rule of Fractions.

But again, as in Whole Mix'd Numbers each Species has a Relation to all the reſt, ſo the ſeveral Members of a Mix'd Whole Number may be expreſſed as Fractions of the higheſt Species; and then if theſe are all added together, they will make with the Number of the higheſt Species, a Mix'd Fractional Number. For Example, 8 *l.* 6 *ſh.* 4 *d.* is equal to *l.* $8 + \frac{6}{20} l : + \frac{4}{240} l.$ and this reduced, is equal to *l.* $8\frac{76}{240}$. And when ſeveral Mix'd Numbers of one kind are thus expreſs'd, the Addition or Subtraction by the Rules of Fractions, will bring out for the Sum or Remainder, a Number equal to what will ariſe from expreſſing the Sum or Remainder got by the common Rules, in the ſame fractional manner with reſpect to the higheſt. And the ſame thing will hold for the Products and Quotes in *Multiplication* and *Division*; for if this were not true, either the Rules given for fractional Operations, or thoſe for Whole Mix'd Numbers muſt be falſe. But each of theſe are demonſtrated to be true.

But again, *Obſerve*, that the common Rules for Mix'd Whole Numbers do make eaſier and diſtincter Work than what would happen by that way of expreſſing the inferiour Species, except upon certain Suppoſitions of their mutual Relations, as we ſhall immediately explain. But keeping to the common Subdiviſions at preſent inſtituted, it is better to expreſs them as Mix'd Whole Numbers, and uſe the Rules given about theſe in the firſt *Book*, and never bring in Fractions when they can be avoided: But this cannot always be done; for ſince Fractions neceſſarily ariſe from Imperfect Diviſion, therefore they will

will unavoidably happen upon the Divifion of Numbers of any loweft Species, or of things for which no inferiour Species are inftituted.

Now, tho' the Operations with Mix'd Whole Numbers, according to the prefent Sub-divifions, are eafier thañ what would be if the Numbers of their inferiour Species were exprefs'd fractionally, and the general Method of Fractions applied; yet there is a Suppofition in the Subdivifion of Quantitys, *i. e.* a certain Species of Fractions, according to which the *Divifion* being made, it would be more eafy and convenient to exprefs the leffer always as Fractions of the greater: and that Species is the *Decimal Fraction* already mentioned; whofe Principles and Operations I fhall firft explain, and then more particularly their Ufe and Application.

C H A P. VIII.

Of D E C I M A L F R A C T I O N S.

§. 1. D E F I N I T I O N.

IF we fuppofe any Integer divided into 10 Parts, and each of thefe again into 10 Parts, making of the Whole 100 Parts; and each of the laft Parts again into 10 Parts, making of the Whole 1000, and fo on: thefe Parts are called *Decimal Parts*; and any Number of them is called a *Decimal Fraction*: Whofe Definition is therefore this, *viz.* a Fraction whofe Den^r is 10, or 10 × 10, or 10 × 10 × 10, &c. *i. e.* 10, or 100, or 1000, &c.

Examp. $\frac{4}{10}$, $\frac{14}{100}$, $\frac{46}{1000}$, &c.

It is plain therefore that the Den^t of any *Decimal Fraction* is 1, with one or more o's on the Right-hand of it. And in this lies the effential Difference betwixt *Decimal Fractions* and all others. But there is alfo another Difference, which is in the *Notation* of them: For tho' they may be written in the Vulgar Form, yet from this Property of the Den^t, we have a Method of Notation different from and eafier than the General or Common Way ufed in the preceding Chapters. And hence alfo we have Operations as Simple and Eafy as thofe of Whole Numbers.

§. 2. N O T A T I O N *of* D E C I M A L S.

THE *Numerator* and *Denominator* of a *Decimal Fraction*, whether Proper or Improper, being known, write firft down the Num^r, then confider how many Cyphers, or o's belong to the Den^t; and beginning at the Right-hand or Place of Units of the Num^r, reckon towards the Left-hand one Figure or Place for every o in the Den^t. And if there are not as many, fupply the Defect with o's, fet on the Left-hand; and fet before them a Point, (reprefenting the 1 belonging to the Den^r) called the *Decimal Point*; which is therefore the Mark of a *Decimal Fraction*: As in the following Examples, written both in the *Vulgar* and *Decimal* Form.

Examp. 1. $\frac{3}{10}$ is .3   *Examp.* 2. $\frac{24}{100}$ is .24   *Examp.* 3. $\frac{426}{10000}$ is .0426

Examp. 4. $\frac{28}{10000}$ is .0028   *Examp.* 5. $\frac{46}{10}$ is 4.6   *Examp.* 6. $\frac{3467}{100}$ is 34.67

This *Notation* is arbitrary, and requires no Demonſtration, but only to ſhew that the Num.r and Den.r are thus diſtinctly marked, tho' not altogether ſeparated from one another: And this is very obvious; for the Num.r is completely expreſſed, (what Cyphers are in ſome Caſes, ſet on the Left-hand of it, changing nothing of its Value,) and becauſe the Den.r conſiſts of a Number of o's on the Right-hand of 1, we want only to know how many are of theſe o's: and this we know by numbering the Places that ſtand on the Right-hand of the Point. Therefore

To read a Decimal which is written in its proper Form ;

Take the whole Rank of Figures, which together make one Number, (*i. e.* the whole Rank excluding the o's that ſtand on the left of all) for the Num.r ; and for the Den.r, reckon as many o's as there are Figures before the Point on the Right-hand. So .057 is $\frac{57}{1000}$, and .004607 is $\frac{4607}{1000000}$.

S c h o l. The ſame *Problems* and *Rules* of *Reduction* belong to Decimals as to any other Fractions, which need not be repeated. There are only theſe few particular things to be remarked, which are conſequences of the Nature and Notation of Decimals, and the general Rules of *Reduction of Fractions* already explained; and which will ſerve as Principles for the Demonſtration of the following Rules of Operation.

C O R O L L A R I E S.

1. A proper Fraction in Decimals can have no Figures ſtanding on the Left-hand of the Point, but all upon the Right ; for the Den.r muſt neceſſarily have more Places than the Num.r, and ſo the Point muſt fall without them on the Left-hand: And an Improper one muſt have Figures on both hands of the Point ; for becauſe the Den.r cannot have more Places than the Num.r, therefore the Point cannot fall without. *Examp.* .046, is Proper ; and 4.62, is Improper.

2. A mix'd Decimal Number and its equivalent Improper Fraction have the ſame *Notation* in Decimal Form, and therefore require no Operation to reduce them ; and ſo the Diſtinction can only be made in the reading them. For Example, $\frac{3467}{100} = 34 + \frac{67}{100}$, have the ſame Decimal Notation, *viz.* 34.67. For this is the Reduction of the Improper Fraction, by the General Rule ; and it is plain the mix'd Number can have no other Notation, except that a Mark of Addition may be put betwixt the Integers and Fraction thus, 34 + .67. But this Mark is ſuperfluous. Hence any Decimal Expreſſion, where there are Figures ſtanding on the Left-hand of the Point, may be read either as an Improper Fraction, or a Mix'd Number ; thus, as an Improper Fraction, taking the whole Rank as it ſtands (without minding the Point) for the Num.r, and for the Den.r reckon as many Cyphers as ſtand on the right of the Point ; or as a Mix'd Number, by taking all the Figures on the left of the Point as a Whole Number, and thoſe on the right as a Decimal Fraction. So this other *Example*, 234.08, is either $\frac{23408}{100}$, or $234 + \frac{8}{100}$.

3. It is manifeſt, that all Expreſſions wherein there are no Figures, but o's after the Point, are pure Whole Numbers compoſed of the Figures ſtanding before the Point. So 34.00 is $= 34$. Yet ſuch Expreſſions are equal to, and may be read as an improper Decimal Fraction, whereof the Num.r is all the Rank of Figures on each ſide the Point, and the Den.r has as many o's as ſtand after the Point ; ſo the preceding is $\frac{3400}{100}$, which,

ac-

according to the Decimal Notation is 34.00. Wherefore a Whole Number is expreſſed in Decimal Form by ſetting any Number of o's after it, and a Point between; which is to be read as a Fraction in the manner explained.

4. Cyphers ſtanding in Places next the Right-hand of a Decimal, make the Fraction the ſame as if they were not there; ſo .3400 = .34. Becauſe theſe Cyphers being reckoned both to the Numr and Denr, if they are taken away, do equally divide both; ſo .3400 being $\frac{3400}{10000}$, divide both Terms by 100, (or take two Cyphers from each) it is $\frac{34}{100}$, or .34.

5. Two Decimal Fractions having the ſame Number of Figures on the right of the Point, have the ſame common Denr; ſo .34 and .06 have the ſame Denr, *viz.* 100. Hence two or more Decimals are reduced to one Denr by adding as many Cyphers on the right of thoſe that have fewer Places, till they have all an equal Number of Places. So theſe, .4, .25, .067, are equal to, .400, .250, .067; having a common Denr 1000.

6. Every proper Decimal Fraction is equal to the Sum of ſo many leſſer ones, whoſe Numrs are the ſeveral ſignificant Figures of the given Numr, their Denrs having as many o's as there are Places from the Point to that Figure. For *Examp.* 1. .34 = .3 + .04, for .3 = .30, which has the ſame Denr with .04; and therefore that Sum is .34. *Ex.* 2. .04608 = .04 + .006 + .00008

§. 3. ADDITION *of* DECIMALS.

R U L E.

WHether the Numbers given are pure or mix'd Decimals, or ſome of them Whole Numbers, write them down under one another in ſuch order that the Decimal Points ſtand all in a Column, and the Figures all in diſtinct Columns, in order as they are removed from the Point either on the Right or Left-hand; then beginning at the Column on the Right-hand, add the Figures in every Column together, and carry forward 1 for every 10 in the Sum, as in Whole Numbers; placing a Point in the Sum under the Points of the given Numbers.

Examp. 1.	*Examp.* 2.	*Examp.* 3.
.24	.004	36.24
.378	.9	450.058
.057	.4067	378.62
.9356	.08	8923.9
.6827	.235	42.007
2.2933	1.6257	9830.825

If ſome of the Numbers given are Whole Numbers without a Fraction, the Work is the ſame, ſetting theſe Whole Numbers on the left of the Column of Points; and if the Sum of the firſt Column in the decimal Part is a Number of 10's, the o need not be written down, but proceed; and do the ſame, if the Figure to be ſet down in the next Column happens alſo to be o. But mind that after a ſignificant Figure comes in the Sum, ſuch o's belonging to the following Columns muſt not be neglected.

Examp.

Examp. 4.	Examp. 5.	Examp. 6.
468.	.456	67.004
24.06	.08	8.206
48.724	37.604	.018
2370.	478.26	.4
8.4	.8	.28
2919.184	517.2	75.908

DEMONST. If you conceive as many o's to be set on the Right-hand of the Decimal Point in each of the Numbers added, till they have all an equal Number of Figures after the Point, they are thereby reduced to Fractions (Proper or Improper) having a common Den^r, (*Corol.* 5.) and the Sum of the Num^{rs} is the Num^r of the Sum sought to be applied to that common Den^r; but thefe o's adding nothing to the Sum, need not be filled up, and therefore the Sum of the Num^{rs} is truly found by the Rule; and by setting a Point in the Sum under the Column of Points in the Numbers added, the common Den^r is right applied, therefore the Rule is good.

Or we may deduce the Reason of this Rule thus: All the Figures of the fractional Parts standing in one Column, are Num^{rs} of Fractions having the same common Den^r, (by *Corol.* 6.) and from the nature of a Decimal Den^r, each 10 in the Sum of any Column is equal to a Fraction whofe Num^r is 1, the Den^r having one o lefs than the Den^r of the Column added; *i. e.* .010 = .01. Therefore it is plain, that adding and carrying of every 10 from the feveral Columns according to the Rule, gives the true Sum.

SCHOL. In *Applicate Numbers*, the Decimals thus added muft all refer to one Integer, or be reduced to that ftate; and in doing this, reduce always the higher Species to the lower, by multiplying the Numerator, or the lower to a higher by dividing the Num^r, if it can be done; and either way the Fraction found will be a Decimal. But if you reduce any other way, the Fraction will not always be a Decimal. *Examp.* .6 Yards + .08 Quarters, are, by Reduction, .24 qr + .08 qr. Or, .6 y^d + .02 y^d. But if inftead of .08 qr, you put .07 qr. then if you reduce this to a Fraction of a Yard, it will not be a Decimal; for it can only be made $\frac{7}{400}$ Yards. And therefore reducing the Higher Denomination to the Lower is the General Rule that will keep the Expreffion in Decimals. But fuch Examples feldom occur; as we fhall fee afterwards in explaining the Ufe of Decimals.

§. 4. SUBTRACTION *of* DECIMALS.

R U L E.

ORder the Subtrahend and Subtractor the fame way as directed in *Addition*, then fubtract as in Whole Numbers, every Figure of the Subtractor from what ftands over it in the Subtrahend, (fuppofing o to ftand where there is no Figure in that Place of the Subtrahend;) and fet a Point in the Remainder, in the fame Column with the Points of the given Numbers; and when o's fall next the Right-hand of the Remainder, they need not be fet down.

Ex. 1.	.84	Ex. 2.	68.28	Ex. 3.	48.	Ex. 4.	478.
	.32		24.057		9.86		.037
	.52		44.223		38.14		477.963

Examp. 5.	82.0642	*Ex.* 6.	234.075	*Ex.* 7.	28.24
	29.7562		82.04		16.
	52.308		150.135		12.24

The *Demonstration* of this Rule stands upon the same Principles as that for Addition. The same *Scholium* is also applicable here, which was made after Addition.

§. 5. MULTIPLICATION *of* DECIMALS.

RULE.

TAKE the Numbers proposed (*i. e.* the Rank of Figures in each) as Whole Numbers, and multiply them one by the other as such, neglecting the o's that stand next the Left-hand, as useless in the Multiplication; then take the Number of Places or Figures, whatever they are, that stand after the Point, both in the Multiplier and Multiplicand, and numbering as many Places from the Right-hand of the Product, set a Point before them; and if there are not as many, supply the Defect with Cyphers, and you have the Product duly expressed; which, according to the Circumstances of the Factors, may be either a Pure Fraction, (*Examp.* 1, 2, 3.) or a Mix'd one, (*Examp.* 4, 5, 6.) or a Whole Number, (*Examp.* 7.)

Examp. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
.46	42.78	.000486	6.08
.28	.00094	.024	.47
368	17112	1944	4256
92	38502	972	2432
.1288	.0402132	.000011664	2.8576

Ex. 5.	*Ex.* 6.	*Ex.* 7.
723.246	578.32	4 56
48.003	64	2.75
2 169 738	231328	22 80
5785 968	346992	3192
28920 84	37012.48	912
34717.077 738		1254.00

DEMONST. The *Reason* of this Rule is obvious; for if you conceive the two given Numbers as Fractions, Proper or Improper, the Work of the Rule is plainly multiplying their Num.rs together, [which are the Rank of Figures proposed, taken as whole Numbers;] and applying to the Product a Decimal Den.r, equal to the Product of the given Den.rs: [Which is done by taking as many Places from the Product of the Num.r for Decimal Places, as there are in the Den.rs of both the Factors; for the Sum of these is the Number of Places in the Product.] And this Fraction is the Product sought, by the general Rule of multiplying Fractions, (*Chap.* 5.)

SCHOL. To multiply any Decimal (Pure or Mix'd) by 10, or 100, &c. (*i. e.* any Number expressed by 1, with o's after it,) it is plain we have no more to do, but remove the Point as many Places towards the Right-hand, as there are o's in the Multiplier; and then, if all happen to be o's on the Left, they are useless and to be neglected.

Examp.

Examp. 1. .46 × 10 = 4.6   *Ex.* 2. .004 × 100 = .4   *Ex.* 3. 82.0375 × 100 = 8203.75

It is plain that this has the fame Effect as the preceding Rule; according to which we fhould fet as many o's after the Multiplicand as there are in the Multiplier; and then taking from the Product as many Decimal Places as are in the Multiplicand, (for we now fuppofe none in the Multiplier) the Point will be removed as many Places to the Right, as the Number of o's annex'd; and thefe being fuperfluous on the Right of a Decimal, need not be fet down. But if there are not as many Places after the Point, the Defect muft be fupply'd with o's, and the Product will be a Whole Number. So 4.6 × 100 = 460. Or the Reafon of this Practice is alfo clear by confidering, that by fetting the Point a Place nearer the Right, every Figure is thereby 10 times the Value it was ; and confequently, fo is the whole.

How to CONTRACT *Multiplication.*

When, betwixt the Multiplicand and Multiplier, there are more decimal Places than we incline to have in the Product, then we may find the Product true to as many decimal Places as we pleafe, (or very nearly true) without producing all the reft of the Figures, which will in many Cafes make a great Abridgement of the Work. For which take this

Rule. Confider how many decimal Places you would have in the Product; fet the Figure in Unit's Place (*viz.* of Integers) of the Multiplier, under that decimal Figure of the Multiplicand, whofe Den^r is what you would have in the Product, (*i. e.* under the 1^{ft}, 2^d, or 3^d, &c. Place after the Point, if you would have only one, two, or three decimal Places in the Product.) Then fet the other Figures of the Multiplier in the reverfe Order from that ; and multiply by every Figure in the Multiplier : In doing which, begin only at the Figure of the Multiplicand, under which the multiplying Figure ftands, neglecting all towards the Right. But at the fame time confider what would have been carried from the Product of the preceding Figures on the Right, (which will be found in moft Cafes by multiplying the two next preceding Figures) that it may be added to the Product which is firft written down. *Again*, Let all the partial Products be fet under one another, fo as the firft Figures in each ftand in one Column, and the reft in the fame Order. *Laftly*, In adding thefe partial Products together, you muft judge as near as you can what would have been carried from the preceding Columns, if we had neglected none of the Figures of the Multiplicand; fo as to add that to the firft Column written down. See the following *Examples.*

Examp. To multiply 47.32685 by 8.463, and have three Places of Decimals in the Product.

Operation at large.	Abridged.
47.32 685	47.32 685
8.463	3 64.8
141 98 055	378 61 4
2 839 61 10	18 93 0
18 930 74 0	. 2 83 9
378 614 80	141
400.527 13 155	400.52 7

As the Allowance for what may be carried from the Columns neglected is altogether a Guefs, we may very often make the Product lefs than it ought to be, by 1 or 2 in the laft Place; which can fcarcely be help'd otherwife than by making one or two more Columns than the Number of Decimal Places you would have in the Product, and then you may cut off the two laft Places from the Product.

Examp. 2. To multiply 463.25639 by 67.864, fo as two Places of Decimals fhall be true in the Product.

Operation at large.	Abridged.
463.25 639	463.2 5 639
67.864	4 6 8.76
1 853 02 556	2779 5 3 83
27 795 38 34	324 2 7 94
370 605 11 2	370 6 05
324 2 794 73	2 7 7 95
27795 383 4	1 8 53
31428.431 65 096	3143 8.4 3

In this *Example* I have fet the 7 which is in the Unit's Place of the Multiplier under the 3^d decimal Place of the Multiplicand, tho' I wanted only two Places in the Product, that by that other Place, the two firft Places may be true.

Obferve, If there be not as many decimal Places in the Multiplicand as are wanted in the Product, you muft fupply them with 'o's. As if in the preceding *Example* the Multiplicand were 463256.39 and I wanted four decimal Places in the Product, then I write the Multiplicand thus: 463256.3900 (or with one o more) and fet the 7 of the Multiplier under the laft o.

Again, If the Multiplier is all a decimal Fraction, imagine a o in the Unit's Place, and fet the other Figures in order from that on the Left.

For the *Reafon* of this Practice it is feen in the Comparifon of the Work at large and abridged, which you fee is but the former reverfed.

§. 6. DIVISION *of* DECIMALS.

R U L E.

TAKE the Numbers propofed as Whole Numbers, *i. e.* fuch a Whole Number as the Rank of Figures would make, without regard to the Point; (in which View o's next the Left-hand will be altogether ufelefs) and as fuch, divide the one by the other. If there is a Remainder, which is neceffarily lefs than the Divifor, fet a Cypher after it, and then divide again: But when the Remainder, with one o added, makes a Number lefs than the Divifor, fet o in the Quote, and add another o; and fo on, till the Remainder, with the o's added, make a Number greater than, (or equal to) the Divifor. And thus continue, adding Cyphers to the Remainder, and dividing, till there be no Remainder. But as this will not happen in every Cafe, the Divifion is to be thus carried on to a greater or leffer Number of Figures, according as the Circumftances of the Queftion require, as fhall be further explained in the ufe of *Decimals.* Obferve alfo, that if at the beginning of the Work, the Dividend makes a leffer Number than the Divifor, when both are confidered as Whole Numbers, then fet as many o's after it, till it be greater than (or equal to) the Divifor; and then begin the Divifion, proceeding with the Remainders, as before directed.

When the Divifion is finifhed, or carried on as far as you think fit, the Quote muft be qualify'd in this manner ; *viz.* Confider how many Decimal Places (or Figures after the Point on the Right-hand) there are in the Divifor, and alfo in the Dividend ; (among which laft are to be reckoned all the o's added to the Dividend, and to the Remainders: and if the Dividend is a Whole Number, the o's added are reckoned the Decimal Places of it.) Then, 1. If the Number is equal in both, the Quote is a Whole Number, (*Ex.* 1.) 2. If the Number in the Dividend is greateft, take the Difference, and feparate as many for Decimal Places from the Right of the Quote, (fupplying the Defect with o's) by a Point fet before them ; and then

then the Quote becomes either a *Proper Fraction* or *Improper*, *i. e.* a Mix'd Number. (*Examp.* 2, 3, *&c.*)   3. If the Number in the Divisor is greatest, take the Difference, and set as many o's after the Quote, and take all for a Whole Number. (*Examp.* 7, 8.)

Examp. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
.004) .128 (32	.32) .152 (.475	.64) .8476 (1.3243	2.7) 109.35 (40.5.
12	128	64	108
008	240	207	135
8	224	192	135
0	160	156	000
	160	128	
	000	280	
		256	
		240	
		192	
		Rem. 48	

Ex. 5.	*Ex.* 6.	*Ex.* 7.	*Ex.* 8.
.8) 2.04 (2.55	4.6) .028 (	.024) .48 (20	.567) 2721.8 (4800
16	or		2268
44	4.6) .0280 (6086		4538
40	276 True Quote,		4536
40	.006086		Rem. 2
40	400		
00	368		
	320		
	276		
	Rem. 44		

Of *Valuing the Remainder, and compleating the Quote.*

There remain yet, as a Part of this Rule, some further Confiderations about the Remainder and Compleating of the Quote. For tho' in the Application and Ufe of Decimals, as we fhall afterwards learn, the Remainder is neglected, yet what I am now to add, is not only fit to be known as a Part of the Theory, but neceffary for our judging aright how far the Divifion ought to be carried on in different Cafes, that the Defect of the Quote, arifing from the Neglect of the Remainder, may not be too great. For this is certain, that where there is a Remainder the Divifion is not perfect; fo that the Quote found, and qualify'd by the preceding Rule, will be deficient of the compleat Quote; and this Deficiency depending on the true Value of the Remainder, we fhall firft fee how that is to be found, and then what is to be added to the Quote already found, to make the compleat Quote.

1. The Remainder is to be valued thus: Make it the Num^r of a Fraction, whofe Den^r is that of the Dividend, (taking in all the Cyphers added in the Operation.) So in *Ex.* 3. the Remainder in its true Value is .000048; in *Ex.* 6. it is .0000044, and in *Ex.* 8. it is .2.

2. If you demand the compleat Quote, (when there is a Remainder) *i. e.* which multiplied by the Divifor will produce the Dividend (refpecting their true Values) you may find it by the general Rule in *Chap.* 6. But if you would keep the Quote already found as one diftinct Part, and would know what is to be added to it that the Sum may be the compleat Quote; then take the Remainder in its true Value (as above,) and divide it by

the Divifor taken alfo in its true Value, by the Rule of *Chap.* 6. and that Quote is the thing fought.   Or thus: Firft make a Fraction of the Remainder and Divifor (as Whole Numbers, without refpecting their true Value) and then fet as many o's after the Den^r as the Decimal Places of the Dividend exceed in number thofe of the Divifor; or if thofe in the Divifor be moft, fet as many Cyphers as the Difference after the Num^r: But if they are equal in Number, you have nothing more to do. And thus you have the thing fought, which will be the fame as that found by the Rule of *Chap.* 6. only in lower Terms, as will eafily appear by comparing them.

Thus in *Ex.* 3. the compleat Quote is $1.3243 + \frac{48}{840000}$; in *Ex.* 6. it is $.006086 + \frac{44}{46000000}$; in *Ex.* 8. it is $4800 + \frac{200}{587}$. See alfo the following *Examples,* where I have only written down the Quotes without the Operation.

Examp. 9. $.23) 46.8 (187 + \frac{5}{25}$.

Ex. 10. $.0432) 342.8 (7900 + \frac{1520}{432}$.

Ex. 11. $.008) 2.68 (330 + \frac{40}{8}$.

Ex. 12. $.08) 2742 (34200 + \frac{600}{8}$.

Obferve again, that this additional Member to the Quote will always be a proper Fraction, when the Number of Decimal Places in the Dividend is equal to, or greater than that in the Divifor; for the Remainder which is the Num^r, is lefs than the Divifor which is the Den^r, and the o's are added to the Den^r: But if it's lefs, then it will be an improper Fraction in fome Cafes, (*Ex.* 10, 11, 12.) which fhews, that the Divifion being carried further on, the integral Number of the Quote would become greater; and particularly if you reduce that improper Fraction, then, as many Figures as its equivalent Whole Number contains, after fo many more Steps in the Divifion, the Quote would have in it all the whole Number that can poffibly belong to it, fo that the additional Member will be after that a proper Fraction: So in *Ex.* 10. the compleat Quote being $7900 + \frac{1520}{432}$, and this Fraction being $= 35\frac{80}{432}$, makes the compleat Quote $7935\frac{80}{432}$. Again, if the additional Member is an improper Fraction, equal to fome whole Number, it fhews, that after fo many more Steps as that Whole Number has Figures, the Divifion would have been perfect without a Remainder. So in *Ex.* 11. the additional Number of the Quote is $\frac{40}{8} = 5$, and the compleat Quote is 335; and in *Ex.* 12. the additional Member is $\frac{600}{8} = 85$, making the compleat Quote 34275.

Wherefore, that the additional Member of the Quote may be always a proper Fraction, (and fo the firft Part never want an Unit of the compleat Quote) carry on the Divifion till the Number of Decimal Places in the Dividend are equal to, or greater than that in the Divifor; unlefs the Divifion is finifh'd without a Remainder before you come to that; for then the Quote found, and qualify'd according to the Rule, is compleat.

DEMONSTRATION *of the preceding* Rule.

The Divifor and Dividend being confidered as Whole Numbers in the Operation, and the o's added to the Dividend and Remainder as belonging to the Dividend; then the Quote being found by the Rule of Whole Numbers, all we have to account for, is the qualifying of the Quote and Remainder, and the additional Member for compleating the Quote. The Reafon of which will eafily appear, by comparing it with Multiplication.

We fhall firft fuppofe there is no Remainder, and then the Product of the Quote and Divifor is equal to the Dividend; but the decimal Places of any Product are equal to the Sum of the decimal Places in the Multiplier and Multiplicand: So the Number of decimal Places of the one Factor is the Difference of the Number in the Product, and in the other Factor, *i. e.* the Number of decimal Places in the Quote, muft be equal to the Difference of the Numbers in the Dividend and Divifor, when the Number in the Dividend is greater, or equal to the Number in the Divifor, (which accounts for thefe two

Cafes,

Cafes, fee *Examp.* 1, 2, 3.) *Again,* When the Number of decimal Places in the Divifor is greater than in the Dividend, then the Quote found (without any Qualification) being multiplied into the Divifor, there would be more decimal Places in the Product than in the Dividend; wherefore that is not the true Quote: But now let as many integral Places of o's as the Difference of the Number of Places in the Divifor and Dividend, be annex'd to the Quote; and this multiplied into the Divifor, there will be the fame Number of decimal Places in the Product as before. But as many of thefe Places next the Right, as the forefaid Difference of decimal Places in the Divifor and Dividend, being o's, becaufe of the o's annex'd to the Quote, they don't increafe the decimal Part; and therefore being cut away from the Product, they leave no more decimal Places in the Product than in the Dividend, the Product being the very fame Rank of Figures: therefore the Quote is truly qualify'd. So if 48 is divided by 24, the Quote is 2; but if the given Numbers are $.024$ and $.48$, the Quote muft be 20, for $.024 \times 2 = .048$, which makes one more decimal Place in the Product than in the Dividend; therefore that is not the true Quote; but reckoning this 20, the Product is $.480 = .48$

In the next place, fuppofe there is a Remainder; then, that the Quote and Remainder are duly qualify'd by the Rule, will eafily appear thus: The Product of the Quote and Divifor is equal to the Dividend, after the Remainder is fubtracted out of it, (taking them all as Whole Numbers;) therefore the Remainder added to that Product, makes the Dividend. But now the Quote being qualify'd, as in the Rule, (*i. e.* with regard to the number of decimal Places in the Dividend and Divifor) the Product muft neceffarily have as many decimal Places as the Dividend, otherways the Quote (which is qualify'd with a Regard to the decimal Places of the Dividend) would not be the true Quote out of that Product; which fhews the Reafon of the Rule for qualifying the Quote in this Cafe. Then for the Value of the Remainder, it's plain its Figures muft be of the fame Value with the Places of the Dividend of which it's the Remainder, which are the laft Places on the Right-hand: Or alfo thus; That the Remainder added to the Product of the Divifor and Quote (in their true Values) may make up the Dividend, it's evident it muft be of the fame Value with the Places of the Dividend to which it's added; and thefe are the laft Places on the Right-hand, which make the Num^r of a Fraction, whofe Den^r is that of the Dividend; wherefore the Remainder muft be fo alfo, (which, according to different Circumftances, will be a Whole Number, or a Fraction, or Mix'd.)

Another DEMONSTRATION.

The *Demonftration* of all that relates to th's Rule, (*viz.* for both Members of the compleat Quote when there is a Remainder) may alfo be eafily deduced from the General Rule in *Chap.* 6. and fhewn to be the fame. *Thus,* When the Divifion is finifhed, or you have put a Stop to it, confider the Dividend and Divifor as Fractions, Proper or Improper, as they happen (reckoning always the o's, added to the Dividend and Remainders, to belong both to the Num^r. and Den^r of the Dividend.) And let $\frac{a}{n}$ be the Divifor and $\frac{b}{m}$ the Dividend, (wherein the n and m are both decimal Den^{rs}, or the one of them fuch, and the other 1, as happens when there is no decimal Place in that Term, but all a Whole Number.) Then by the General Rule, the compleat Quote of $\frac{b}{m}$ divided by $\frac{a}{n}$, is $\frac{bn}{am}$ equal to $\frac{b}{a} \times \frac{n}{m}$. Now $\frac{b}{a}$ is the Quote of the two Num^{rs}, which if it's a Whole Number (there being no Remainder) call it q, and if there is a Remainder, let it be r; then is $\frac{b}{a} = q + \frac{r}{a}$ (the very thing found by the Rule of Divifion of Decimals:) But

this

this muſt be multiplied by $\frac{n}{m}$, *i. e.* both Parts, q, and $\frac{r}{a}$, if there is a Remainder; which is the ſame very thing in effect that this Rule directs to be done, for qualifying the Quote firſt found, and then compleating it.   For it's plain, if $n = m$ (*i. e.* if the decimal Places in the Diviſor and Dividend are equal in number) the Quote found is not thereby al-ter'd, and therefore $q + \frac{r}{a}$ is the compleat Quote.

Again, Suppoſe n and m unequal, then by cutting away an equal Number of o's from n and m, they are equally divided.   Now therefore if m (the Den^r of the Dividend) has more o's than n (the Den^r of the Diviſor) the Fraction $\frac{n}{m}$ is equal to a Fraction whoſe Num^r is 1, and its Den^r a decimal one, having as many o's as the Difference of the Num-ber of o's in n and m.   For *Examp.* $\frac{n}{m} = \frac{10}{1000} = \frac{1}{100}$.   But to multiply by ſuch a Frac-tion, is to divide by its Den^r; *that is,* plainly to divide $q + \frac{r}{a}$ (the Number found by the Operation) by a decimal Den^r, having as many o's as the decimal Places of the Di-vidend exceed in Number thoſe of the Diviſor.   And this Diviſion is done by ſetting off ſo many decimal Places in the Part q; and in the Part $\frac{r}{a}$, by ſetting as many o's after the Den^r a, which is the 2d Caſe in the Rule.

Again, let n be greater than m, then is $\frac{n}{m}$ equal to a Whole Number conſiſting of 1, and after it as many o's as thoſe in n exceed thoſe in m.   *Examp.* $\frac{n}{m} = \frac{1000}{10} = 100$. But $q + \frac{r}{a}$, is to be multiplied by this; which is done by ſetting as many integral Places of o's both after q, and after r, as there are in $\frac{n}{m}$ ſo reduced; *i. e.* as the decimal Places of the Diviſor exceed in Number thoſe of the Dividend.   Which is the laſt Caſe of the Rule.

We have here demonſtrated the Reaſon for qualifying the Quote, and compleating it; but have not conſidered the true Value of the Remainder by itſelf, which is that in which it muſt be added to the Product of the Diviſor and Quote, taken in their true Values to make up the Dividend, (according to its true Value.)   But this is done in the former Demonſtration.

Of the Uſe *and* Application *of* Decimal Fractions.

WE have already obſerved, that the great Benefit propoſed by Decimal Fractions, is a more ſimple and eaſy Operation than what Vulgar Fractions, taken either in their proper Form, or as mix'd Integers, do require.   We ſhall conſider how the Application is made for anſwering that End, and how far it's a real Advantage.

In the firſt place, this is very evident, that if inſtead of the Subdiviſion of Coins, Weights and Meaſures, (and other kind of Quantities uſeful in Society) which now obtain, there were one ſtandard ſuperior Species, and all the Subdiviſions were Decimals, whe-ther the ſeveral Parts were alſo diſtinguiſhed by Names, or only by their decimal Deno-minations, it were the ſame thing to the purpoſe; then the Common Operations would be as ſimple and eaſy as Whole Numbers.   The Rules and Reaſons of which are, I hope, compleatly explained in the preceding Part.   But ſuppoſing this were ſo, yet either we could not entirely avoid the Conſideration of Vulgar Fractions, or we muſt admit of ſome

In.

Inaccuracies in Calculations, which are unavoidable with Decimals; and which will be of more or less Consequence in different Circumstances. For we have seen that Decimals will have Remainders (because every Number is not an Aliquot Part of every other) and then the Quote is not compleat without bringing in a Vulgar Fraction; and therefore if we take the Quote without this Correction, it's less than just according to the Value of the Remainder, or rather the Value of the Vulgar Fraction that's necessary to compleat it. Now, if the Number found by this Division is the final Answer of a Question, which is to be applied in no further Calculation, then if it is brought so low as to be less than any Quantity of that kind that is used, (for Example, the smallest real Coin or Weight, &c. that has any Name or distinct Being in Society) then the Defect is not to be complained of; because if you do compleat the Quote, the additional Part is of no use: But if a Quote is to be further employ'd in Calculation, especially if it's to be multiplied, the Defect may become considerable; and it will be the more so, as the Multiplier is greater, and also according to the Value of the Integer. Now the only Remedy for this, while we use none but decimal Fractions, is to bring the Division very low ; *i. e.* carry it on till the Den^r be very large, and consequently what is deficient be very little: and this is to be regulated according to the Circumstances above-mentioned; for which you'll find more particular Rules afterwards. But then this Inconveniency will frequently happen, That by this means we shall have very large Numbers to work with, which will prove more troublesome than the Method of Vulgar Fractions. These things we shall find more particularly exemplify'd afterwards.

Again, Tho' decimal Subdivisions are not in common use, yet they may be applied by a Reduction of the common Species to Decimals, and these back again to the other: I shall therefore explain this Reduction, and then by particular *Examples* shew the Application, with such Remarks as may give a general View of the Conveniences and Inconveniences of Decimals, and consequently help to judge where they are preferable or not to the common Method.

PROBLEM I.

To reduce a Vulgar Fraction to a Decimal.

R U L E. To the Num^r of the given Fraction add one or more o's, as Decimal Places, till it be greater than (or equal to) its Den^r; then divide by the Den^r, adding o's to the Remainders, and carrying on the Division (as directed in *Division of Decimals,*) till o remain, or as far as you please: then make the Quote a Num^r, and apply to it a Decimal Den^r, with as many o's as the Number of o's added to the Dividend and Remainders. This Decimal is exactly equal to the given Vulgar Fraction, if there is no Remainder in the Division; but if there is still a Remainder, that Decimal is deficient by a Compound Fraction, the one Member of which is a Simple Fraction whose Num^r is the Remainder, and its Den^r the Divisor; and the other Member is a Fraction whose Num^r is 1, and its Den^r is that of the Decimal already found. Or it is a Simple Fraction whose Num^r is the Remainder, and its Den^r the Divisor multiplied by the Den^r of the Decimal Fraction found; *i. e.* having as many o's prefix'd to it as belong to the Den^r; wherefore the more Places the Decimal found has, the less is the Defect.

The *Reason* of this Rule you have in the *Division of Decimals*; for the Dividend with the o's added is an Improper Decimal. Or you may take it from *Prob.* 7. *Chap.* 2. for the o's added to the Num^r and Remainder belong to the Decimal Denominator of the Fraction sought, by which the Num^r of the given Fraction is multiplied, and the Division made finds the correspondent Num^r according to that Rule, to which the Den^r is ap-

plied

plied according to the Notation of Decimals; and what this Decimal Fraction is deficient, is also found according to the same Rule.

Ex. 1. $\frac{1}{2} = .5$ *Ex.* 2. $\frac{3}{4} = .75$ *Ex.* 3. $\frac{1}{16} = .0625$ *Ex.* 4. $\frac{7}{13} = .5384,$ &c.

Operation.	Oper.	Oper.	Oper.
2) 10 (5	4) 30 (75	. 16) 100 (625	13) 70 (5384
	28	96	65
20	――	――	――
20	20	40	50
――	20	32	39
	――	――	――
		80	110
		80	104
		――	――
			60
			52
			Rem. 8

In the 4th *Examp.* there is a Remainder 8; and so the Quote wants $\frac{8}{110000}$ of the complete Value of the given Fraction; *i. e.* $\frac{7}{13} = .5384 + \frac{8}{110000}$, this being the true Value of the Remainder.

Observe, The Decimal may be found all at once in the Division, by first setting a Point in the Place of the Quote; then if one o does not make the Num^r equal to the Den^r, set o after the Point; and if another o makes it still less than the Den^r, set another o in the Quote, and so on: *That is,* set as many o's after the Point in the Quote as the Number of o's, which being set on the Right-hand of the Num^r, leaves it still less than the Den^r; and then after these o's comes the Num^r of the Decimal, found by setting one o more on the Right-hand of the Num^r which gives the first significant Figure of the Quote, and by o's gradually annex'd to the Remainders, the Work is carried on, and the Num^r and Den^r of the Decimal sought are thus both together formed; the Reason of which is manifest from the way of finding and applying the Den^r to the Num^r. See this *Example.*

Ex. $\frac{1}{128} = .0078125$

Operation.
128) 1000 (.0078125
896
――――
1040
1024
――――
160
128
――――
320
256
――――
640
640
――――

COROLLARY I. In dividing any Whole Number by another, when there is a Remainder, instead of making a Vulgar Fraction of it, we may turn it into a Decimal equal or nearly equal to it, by carrying on the Division with o's added to the Remainders, (in the manner taught in *Division of Decimals,* or in the preceding *Problem,*) till o remain; then the Decimal is equal to the Vulgar Fraction: or till there be many Decimal Places, and then it is nearly equal to it. But how far it ought to be carried on, depends upon Circumstances of the Application   (See the following *Scholium* 2.) And after the Integral Quote found, set a Point; so that all the Figures that come after, are Decimal Places: As in the following *Example.*

```
28 ) 7645 ( 273.0357
     56
     ----
     204
     196
     ----
      85
      84
     ----
      100
       84
     ----
      160
      140
     ----
      200
      196
     ----
Rem.   4  whofe
```
true Value is   .0004.

COROL. 2. When of a fimple applicate Whole Number it is propofed to find a certain Part, inftead of reducing the Remainders in the Divifion to lower known Species, we may carry on the Divifion decimally; and fo all the Numbers of inferiour Species that would arife by reducing and dividing are thus turned into a decimal Fraction, (the Defign and Ufe of which you will hear of more particularly afterwards.) If the Divifion does not foon come to an end, carry it on as Circumftances make it neceffary. See the Rule in *Schol. 2.* following.

SCHOLIUM I.

That every Vulgar Fraction is not reducible to a determinate Decimal, (*i. e.* where there is no Remainder in the Divifion,) we know in fact by Examples; in which we find this certain Mark That the Divifion will never come to an end, *viz.* that there happens a Remainder which is the fame with a former Remainder; in which Cafe it is not only certain that the Divifion will never have an end, but this we know alfo that the remaining Figures of the Quote muft neceffarily be a continual Repetition of the fame Figures (in the fame order) that ftand already in the Quote from that one which proceeded from that former Remainder; as in thefe Examples.

Examp. 1.
```
3 ) 20 ( 66, &c.
    18
    ----
    20
    18
    ----
    2, &c.
```

To reduce ⅔ to a Decimal, it is .666, *&c.* (the 6 being always repeated,) for it is manifeft the fame Figure 6 will always arife in the Quote, becaufe it is the fame Dividend 20.

Examp. 2. To reduce $\frac{211}{495}$, it is .42626, *&c.* the 26 being ftill repeated; for the Remainder at which the Operation is ftop'd being the fame as a former which was the very firft Remainder, it is plain that carrying on the Work, we fhould have for the next two Figures in the Quote 26, and fo on ftill 26 *in infinitum.* Therefore whenever this happens in any Cafe, we need proceed no further, but obferving what Figures in the Quote would be repeated, take as many of them, or the whole of them, as many times as we think fit.

Operation of Ex. 2.
```
495 ) 2110 ( .426, &c.
      1080
      ----
      1300
       990
      ----
      3100
      2970
      ----
Rem.   130
```

Such Decimals are very properly called Circulating Decimals, becaufe of the continual Return of the fame Figures; and may be called Indeterminate or Infinite Decimals, becaufe they can never come to an end : as we alfo call thofe which are the Effect of a Reduction which has no Remainder, Finite or Determinate Decimals. *Obferve* alfo, that thefe Infinite Decimals may be reckon'd as complete, becaufe tho' they are compofed of an infinite Series of Fractions, yet there is a certain and known Order in the Progreffion of the Series, from the conftant Repetition of the fame Figures, whereby

whereby they are capable of being managed in Operations fo as nothing fhall be wanting. But the Demonftration of the Theory and Rules of Operation with fuch Fractions requires other Principles than have yet been explained, and muft therefore be referred to another place: (See *Book* 5. *Chap.* 4.) wherein you'll find it demonftrated, that every Vulgar Fraction will reduce to a Decimal, either finite or circulating.

Scholium II.

Tho' fome Vulgar Fractions will become finite Decimals, yet if thefe have a great many Places, the Ufe of them will become very inconvenient and tedious in Practice. Alfo, tho' we have Rules for managing Circulating Decimals without any Defect, yet the fame Inconveniency will arife in thefe, when they circulate upon many Figures, or when the Circulation begins at a great diftance from the Point; therefore it is fufficient for common Ufe to carry the Reduction fo far only, as that the Defect be inconfiderable; (for the further the Reduction is carried, the Defect is the lefs:) in order to which, I fhall here fhew you how

PROBLEM II.

To carry the Reduction of a Vulgar Fraction fo far, that the Decimal found fhall want lefs than any affigned Fraction.

RULE. Let the affigned Fraction be reprefented by $\frac{a}{b}$, if a Decimal is carried to fo many Places after the Point, as are expreffed by b; the defect of that Decimal cannot be equal to $\frac{a}{b}$: For it cannot want a Fraction whofe Num^r is 1, and its Den^r that of the decimal Quote already found, which we may exprefs $\frac{1}{100\,\&c.}$ the Den^r having as many o's as b has Figures. Since in that Cafe the Figure in the laft Place found would neceffarily be greater than it is by 1; or, becaufe by the preceding Rule of this *Problem* the defect is only a compound Fraction, whereof one Member is this Fraction $\frac{1}{100\,\&c.}$ which Defect is therefore lefs than $\frac{1}{100\,\&c.}$ and this is evidently lefs than $\frac{a}{b}$; fince 100, $\&c.$ having as many o's as b has Figures, muft be a greater Number; and a is not lefs than 1 : Wherefore $\frac{1}{100\,\&c.}$ muft be lefs than $\frac{a}{b}$.

To apply this more particularly : Confider to what Integer any Decimal refers; reduce that Integer to the loweft known Denomination; if the Decimal has as many Places after the Point, as that Number of the loweft Denomination which is equal to the Unit to which the Decimal refers, then the Decimal does not want the Value of an Unit of that loweft Denomination. And if the propofed Decimal were again to be multiplied by any Number, then to make it fo that the Product fhall not be deficient by an Unit of the loweft Denomination, make it have as many Places after the Point, as the Sum of the Number of Figures in the propofed Multiplier, and in the Number of Units of the loweft Denomination which makes an Unit of the Denomination to which the Decimal refers.

For *Examp.* If a Decimal of 1 *l.* has 3 Places after the Point, it does not want $\frac{1}{1000}$ of it, therefore does not want 1 Farthing, which is $\frac{1}{960}$ of it. And if the Decimal is again to be multiplied by a Number of 5 Places, let the Decimal have 8 ($= 5 + 3$) Places, and the Product fhall not want 1 *f.* For being carried to 8 Places, it cannot want $\frac{1}{100000000}$ of 1 *l.* which is $\frac{1}{100000}$ of $\frac{1}{1000}$. But $\frac{1}{1000}$ is lefs than 1 *f.* and

2

hence

hence $\frac{1}{100000}$ of $\frac{1}{1000}$ *l.* is lefs than $\frac{1}{100000}$ of 1 *f.* Confequently if the Decimal carried to 8 Places is multiplied by a Number of 5 Places, (which is lefs than 100000) the Product cannot want 1 *f.* The Univerfality of this Reafon for all Cafes is manifeft.

PROBLEM III.

To reduce Integral Numbers of inferiour Denominations to the Decimal of a higher.

CASE I. *To reduce a Simple Number.*

RULE. Exprefs it firft as a Vulgar Fraction of the higher, by making itfelf the Num^r, and taking for the Den^r the Number of the inferiour Denomination that is equal to 1 of the higher. Then reduce this Vulgar to a Decimal Fraction, by the laft *Problem*.

Examp. 1. To exprefs 5 *fb.* in the Decimal of a Pound: Firft, it is $\frac{5}{20}$ *l.* and this again is reduced to .25 *l.*

Operation.

$$112) 300 (.0267857$$
$$224$$
$$\overline{760}$$
$$672$$
$$\overline{880}$$
$$784$$
$$\overline{960}$$
$$896$$
$$\overline{640}$$
$$560$$
$$\overline{800}$$
$$784$$
$$\text{Rem. } \overline{16}$$

Examp. 2. 3 ℔ *Averdupoife* Weight the greater, to the Decimal of a hundred Weight. It is $\frac{3}{112}$ *C.* and this is again .02678 *C.* Here the *Divifion* is imperfect, and we are to carry it on lefs or more as Circumftances require, according to the preceding Directions.

Another Method.

℔.

$$28 \begin{cases} 4 | 3.00 \\ 7 | .7500000 \\ 4 | .1071428 \\ .0267857 \end{cases}$$

It will be in moft Cafes eafier to divide gradually from one Species to another, as in the Margin ; where 3 ℔ is divided by 28, (at two Steps, *viz.* by 4 and 7.) to bring it to the Decimal of 1 *qr.* and this again by 4, which brings it to the Decimal of 1 *C.*

CASE II. *To reduce a Mix'd Number.*

RULE. Reduce each of the Numbers by the firft *Cafe*, and then add their Decimals together : Or reduce the mix'd Number to a fimple Number of the loweft Species, and then turn that into a Decimal.

Examp. To reduce 9 *fb.* 6 *d.* to a Decimal of 1 *l.* it is .475 *l.* by adding .45, (the Decimal equal to 9 *fb.*) and .025, (that equal to 6 *d.*) or by reducing 9 *fb.* 6 *d. viz.* 114 *d.* equal to $\frac{114}{240}$ *l.* = .475 *l.*

Another Method to find the Decimal of a mix'd Number. Reduce the Number of the loweft Species to the Decimal of the next above, (whether there be any Number of that Species in the Queftion, or not,) add to it the Number of that Species in the Queftion,

(it

(if there is any) and reduce the Sum to the next higher Species; adding to the Number found, the Number of that Species given in the Queſtion; and go on ſo till you come to the propoſed Integer.

Examp. To reduce 4 *ſb.* 7 *d.* 3 *f.* to the Decimal of a Pound: Makes .232291 6 *l.*

4	3.00
12	7.750000
20	4.6458333, &c.
	.2322916, &c.

Thus, 3 *f.* is .75 *d.* to which add 7 *d.* Then is 7.75 equal to .6458333, &c. *ſb.* to which add 4 *ſb.* and then is 4.645833, &c. equal to .232291666, &c. *l.*

SCHOLIUM.

Concerning the Conſtruction and Uſe of DECIMAL TABLES.

In order to the Application of Decimals, we ought to have ready calculated the Decimal of any Integer of *Money, Weight, Meaſure,* &c. anſwering to every Number, Simple or Mix'd, of inferiour Denomination, and of leſs Value than that Integer; which Decimals being orderly collected and diſpoſed, make what we call *Decimal Tables,* by which any Decimal required may be readily found, or alſo the Value of any given Decimal in known inferiour Species.

As to the Conſtruction of theſe Tables, it would certainly be a very tedious Work to find every Decimal by a ſeparate Application of this *Problem,* [tho' this is a complete general Rule.] There are other Methods to ſhorten and make that Conſtruction eaſier; which I ſhall here explain as far as the Principles already taught do permit.

RULE. Find the Decimal equal to an Unit of the loweſt Species; and if that is a determinate Decimal, then from it, as the Root, the Decimals of all the reſt may be found accurately by *Addition.* Thus, double the Root, that gives the decimal Fraction for 2 of that loweſt Species; then add the Decimal of 1 and 2, the Sum is the Decimal of 3, and ſo on by adding ſtill the laſt Decimal to the firſt, till you come to a Number of that Species equal to an Unit of the next Species above; then make that the Firſt or Root of all the Decimals of that Species, making them up the ſame way as the laſt, *i. e.* doubling the firſt for the Decimal of 2 (of that Species,) add the Decimal of 1 and 2 for 3, and ſo on in this manner go through all the Species till you come to the Integer itſelf; and if you add the Decimal of the Number next leſs than the Integer to the Root, the Sum will be Unity in the Place of Integers.

If the Root, carried to a certain Number of Places, is not determinate, then is it a deficient Decimal; and if we make up the Table from that Root, all the other Decimals in the Table are alſo deficient, wanting gradually more and more from the Root upwards, ſo that the Number that comes againſt the Integer will be a Decimal. But the more Places the Root is carried to, the leſs is the defect in that and every other Part of the Table. And that there may not be wanting in any of theſe Decimals an Unit of the loweſt Species, or any Part you pleaſe of ſuch an Unit, follow the Directions already given.

Obſerve again, That as all Vulgar Fractions become Decimals, which are either determinate or circulate, ſo there is an eaſy way of making up the Tables by Addition from a Root which has one or more circulating Figures, (and that by uſing only the firſt Period of them) ſo as all theſe Decimals which would be found determinate by a ſeparate Reduction, ſhall come out ſo in the Table, and all the reſt circulate upon the ſame Number of Figures in the ſame Places as the Root does. But as the Reaſon of this Method de-

* X

de-

depends upon the particular Doctrine of circulating Decimals, and both the Method and Reason will be easily understood when you learn that Doctrine in *Book* 5. *Chap.* 4. I shall say no more of it here. And only *observe* these few things.

1. That to bring some Vulgar Fractions to a Decimal, determinate or circulating, will be such a long Work, that it's more convenient to take it imperfect with a less number of Places.

2. As to the following Tables *observe*, That the Roots of some of them are determinate; whence all the other Decimals of such Tables are also determinate; and are known by their wanting this Sign + after them. In others the Root circulates, and so the Table is made up not by the Method of Addition simply, but with a due regard to that Circulation; so that the Decimals in the Table are these which would be found by calculating each separately, and carrying it to the same Number of Places.

3. But again *observe*, That the Roots of some of the Tables circulating upon a single figure, that Decimal is taken as far as the first Figure of the Circulation; and what Decimals in such Tables do circulate, are marked by the Sign + after the circulating Figure; those which want it being determinate.

4. In the last place *observe*, That for the Tables of *Averdupoise Weight the greater*, and of *Time*, the Root was carried on to a Circulation, and the Table made up with a regard to that: But these Decimals running to many Places, you have here only the first seven Places, which are enow for common Use. And for *Money*, I have made two different Tables; the Manner and Design of which I have explained with the Table.

DECIMAL TABLES of MONEY, WEIGHTS, *and* MEASURES.

TABLE I. MONEY.

The Integer 1 Pound.				Another TABLE of MONEY.		
Farth.	1	.0010416	┿	Farth.	1	.0010416666
	2	.002083	┿		2	.0020833332
	3	.003125			3	.0031249998
Penny	1	.00416	┿	Penny	1	.0041666664
	2	.0083	┿		2	.0083333328
	3	.0125			3	.0124999992
	4	.016	┿		4	.0166666656
	5	.02083	┿		5	.0208333320
	6	.025			6	.0249999984
	7	.02916	┿		7	.0291666648
	8	.03	┿		8	.0333333312
	9	.0375			9	.0374999976
	10	.0416	┿		10	.0416666640
	11	.04583	┿		11	.0458333304
Shilling	1	.05		Shilling	1	.0499999968
	2	.1			2	.0999999936
	3	.15			3	.1499999904
	4	.2			4	.1999999872
	5	.25			5	.2499999840
	6	.3			6	.2999999808
	7	.35			7	.3499999776
	8	.4			8	.3999999744
	9	.45			9	.4499999712
	10	.5			10	.4999999680
	11	.55			11	.5499999648
	12	.6			12	.5999999616
	13	.65			13	.6499999584
	14	.7			14	.6999999552
	15	.75			15	.7499999520
	16	.8			16	.7999999488
	17	.85			17	.8499999456
	18	.9			18	.8999999424
	19	.95			19	.9499999392
					20	.9999999360

Obferve, This fecond *Table of Money* I have made for no other end but to fhew, that if the Root of any Table is not determinate, yet being taken to many Places, the Error will be very little; for the Table being here carried to 20 *fh.* this inftead of being equal to 1 *l.* [as it would be had the Root been determinate, or the Numbers calculated with a due regard to that, as in the firft Table,] it is a Decimal of 1 *l.* but which wants lefs than ₁₀₀₀₀₀₀₀ Part; the reft wanting lefs as they ftand nearer to 1 *f.* For the Error grows from 1 *f.* to 1 *l.*

TABLE

TABLE II. *Troy Weight.*

The Integer		1 ounce.	
Grains	1	.002083	+
	2	.00416	+
	3	.00625	
	4	.0083	+
	5	.010416	+
	6	.0125	
	7	.014583	+
	8	.016	+
	9	.01875	
	10	.02083	+
	11	.022916	+
	12	.025	
	13	.027083	+
	14	.02916	+
	15	.03125	
	16	.03	+
	17	.035416	+
	18	.0375	
	19	.039583	+
	20	.0416	+
	21	.04375	
	22	.04583	+
	23	.047916	+
Penny Weight	1	.05	

The Table for Penny Weights is the same as that for Shillings with respect to 1 Pound.

TABLE V. *Of Liquid Measure.*

The Integer		1 gall.
q^r of a Pint	1	.03125
	2	.0625
	3	.09375
Pint	1	.125
	2	.25
	3	.375
	4	.5
	5	.625
	6	.75
	7	.875

TABLE III. *Averdupoife Weight the Leffer.*

The Integer		1 ℔.	
q^r of a Dram	1	.00097656	+
	2	.00195313	+
	3	.00292969	+
Dram	1	.00390625	
	2	.0078125	
	3	.01171875	
	4	.015625	
	5	.01953125	
	6	.0234375	
	7	.02734375	
	8	.03125	
	9	.03515625	
	10	.0390625	
	11	.04296875	
	12	.046875	
	13	.05078115	
	14	.0546875	
	15	.05859375	
Ounce	1	.0625	
	2	.125	
	3	.1875	
	4	.25	
	5	.3125	
	6	.375	
	7	.4375	
	8	.5	
	9	.5625	
	10	.625	
	11	.6875	
	12	.75	
	13	.8125	
	14	.875	
	15	.9375	

TABLE IV. *Averdupoife Weight the Greater.*

The Integer		1 Cwt.
q^r of an Ounce	1	.0001395
	2	.0002790
	3	.0004185
Ounce	1	.0005580
	2	.0011160
	3	.0016741
	4	.0022321
	5	.0027901
	6	.0033482
	7	.0039062
	8	.0044642
	9	.0050223
	10	.0055803
	11	.0061383
	12	.0066964
	13	.0072544
	14	.0078125
	15	.0083705
Pound	1	.0089285
	2	.0178571
	3	.0267857
	4	.0357142
	5	.0446428
	6	.0535714
	7	.0625, Exact.
	8	.0714285
	9	.0803571
	10	.0892857
	11	.0982142
	12	.1071428
	13	.1160714
	14	.125, Exact.
	15	.1339285
	16	.1428571
	17	.1517857
	18	.1607142
	19	.1696428
	20	.1785714
	21	.1875, Exact.
	22	.1964285
	23	.2053571
	24	.2142857
	25	.2232142
	26	.2321428
	27	.2410714

TABLE

T A B L E VI. *Of Dry Meafure.*

The Integer		1 qr Chald.
Pint	1	.001953
	2	.003906
	3	.005859
q^r of a Peck {	1	.0078125
	2	.015625
	3	.0234375

For Pecks and Bufhels, they have the fame Decimals as q^r of Pints, and Pints in Liquid Meafure with refpect to 1 Gallon.

T A B L E VII. *Of Long Meafure.*

The Integr.		1 yd.
q^r of Nails	1	.015625
	2	.03125
	3	.046875
Nails	1	.0625
	2	.125
	3	.1875
q^r of Yard	1	.25
	2	.5
	3	.75

T A B L E VIII. *Of Long Meafure.*

The Integer		1 foot.	
q^r of an Inch {	1	.02083	+
	2	.0416	+
	3	.0625	
Inches	1	.083	+
	2	.16	+
	3	.25	
	4	.3	+
	5	.416	+
	6	.5	
	7	.583	+
	8	.6	+
	9	.75	
	10	.83	
	11	.916	+

T A B L E IX. *Of Time.*

The Integer		1 Year.
Days	1	.0027397
	2	.0054794
	3	.0082191
	4	.0109589
	5	.0136986
	6	.0164383
Weeks	1	.0191780
	2	.0383561
	3	.0575342
	4	.0767123
	5	.0958904
	6	.1150684
	7	.1342465
	8	.1534246
	9	.1726027
	10	.1917808
	11	.2109589
	12	.2301369
Quarter	1	.25
	2	.5
	3	.75

13 Weeks	.2493150
26	.4986301
39	.7479452
52	.9972602

In this Table of Time, the Decimal for 1 Day is taken by the 365th Part of a Year; and fo the Table is carried on to 12 Weeks, or 84 Days. Then as I reckon 13 Weeks to 1 Quarter of a Year, fo after 12 Weeks comes next 1 Quarter. But that Decimal and the following are taken accurately, which would not happen if they were continued from the preceding. So that as the Decimals from 1 Day to 84, or 12 Weeks, are true to 7 Places; thefe for Quarters are accurate, tho' a Quarter is not a precife Number of Days, but 91 Days and $\frac{1}{4}$ of a Day; reckoning 365 Days to a Year: So that in applying this Table to the Calculations of Intereft, for which chiefly it is defigned, you muft reckon 91 Days to a Quarter; by which means the Decimal will be a little greater than what correfponds to 91 Days. But if we continue the Table, then the Decimals for 13 Weeks, 26 W. 39 W. 52 W. (which are lefs than 1 Year by 1 Day,) are as in the Margin. And if we reckon 13 Weeks a Quarter, then thefe are the Decimals for 1, 2, 3, 4 Quarters; but deficient for the exact Quarter of a Year, &c. And perhaps it may be beft to ufe thefe Numbers. So that if the time is within 13 Weeks, or 91 Days, take the Decimals in the 2 upper Parts of the Table; and if it exceeds 91 Days, take for 91 Days (= 13 W.) or 182 Days (= 26 W.) or 273 Days (= 39 W.) the Decimals in this laft Part; taking Decimals for what Days are over any of thefe Numbers, and lefs than 91, in the former Part; and add all together. In fhort, Reduce the Number of Days to Quarters, Weeks, and Days, (at 91 Days to a Quarter,) and take their correfponding Decimals and add together. You fhall fee the Application particularly afterwards.

U S E

USE of the preceding TABLES.

I. To find the Decimal of any Integer (which is in the Tables) anfwering to any Number, fimple or mix'd, lefs than that Integer.

Rule. (1.) If it is a fimple Number, feek it in the Left Column of the Table relating to that Integer, and againft it you have the Decimal fought: So for 7 *s.* (in the Table of Money) we find .35 *l.*

(2.) If it is a Mix'd Number, feek the Decimals anfwering to the feveral Members; their Sum is the Decimal fought. *Examp.* The Decimal for 9 *s.* 8 *d.* is .483 (the 3 circulating for ever) which is the Sum of .45 and .0333*f.* the Decimals of 9 *s.* and of 8 *d.*

Obferve. Decimal Tables would be more compleat, if they were made up for every Number, Mix'd as well as fimple, lefs than the Integer; but as this would fwell them to a great Bulk, fo thefe for the fimple Numbers are fufficient; becaufe from them the others can be got eafily as there is occafion. Or if any body wants fuch Tables, they are eafily made, either by adding their Parts, or by the Method of *Cafe* 2. *Probl.* 3.

II. Having the Decimal of any Integer to find the correfponding Number, fimple or mix'd, of known inferior Species.

Rule. (1.) Seek the given Decimal in the Table; if you find it there, againft it ftands the Number fought: So againft .75 in the Table of Money ftands 15 *s.*

(2.) If the given Decimal is not exactly found in the Table, take the next leffer found there, the Number againft it is Part of the Anfwer; then take the Difference betwixt that Decimal and the given one; and feek it or the next leffer in the Table, and againft it you have another part of the Anfwer in a lower Species than the preceding part. Go on thus as long as you can, and you'll find the Anfwer as near as poffible in known Species.

Examp. 1. To find the Value of .6875 *l.* I feek this in the Table, but the neareft to it (leffer) which I can find is .65, to which anfwers 13 *s.* then the Difference of .6875 and .65 is .0375, which I find in the Table, againft 9 *d.* therefore the Anfwer is 13 *s.* 9 *d.*

Examp. 2. For .4768 *l.* the neareft leffer Decimal is .45 againft 9 *s.* then .4768 lefs .45 is .0268, and the next lefs than this is .025 againft 6 *d.* then .0268 lefs .025 is .0018, the next lefs than which is .00104, &*c.* againft 1 *farth.* So the Anfwer is 9 *s.* 6 *d.* 1*f.* with a Fraction of a Farthing.

But *obferve,* That if any Decimal given is not found exactly in the Tables, the Value of it may be had in moft cafes as eafily, by Reduction (as in *Probl.* 12. Reduction of Vulgar Fractions.) And for any Integer in the preceding Tables, it will be fufficient to take the firft three Places after the Point. But the eafieft way to folve the Problem is by fuch compleat Tables as are already mentioned.

As the *Decimals* of Money are of the greateft ufe, fo alfo there is an eafy way of finding the Value of any Decimal of a Pound; or finding the Decimal for any Number lefs than 1 *l.* without Tables. Thus:

I. *To find the Value of any Decimal of* 1 *l. without Tables or Pen.*

Rule. Take the firft three Figures after the Point, neglecting the reft; then double that Number which ftands in the firft Place (after the Point) it is fo many Shillings of the Anfwer. And if the Figure in the fecond Place is 5 or greater, add 1 to the Shillings already found; then take what the Figure in the fecond Place exceeds 5, with the Figure in the third Place, (and if there is no Figure in the third Place, fuppofe o) confider thefe two Figures, in order as they ftand, as one Number. If they make a Number not exceeding 23, take fo many Farthings (and reduce them to Pence) for the remaining Part of the Anfwer.

I

Anfwer: But if that Number exceeds 23, take 1 from it, and the Remainder is fo many Farthings in the Anfwer. Thus you fhall have the Anfwer exact in all its Value of known Species; or fo, that the Error fhall not be 1 Farthing.

Examp. 1.	$.4 l. = 8 s.$	*Examp* 5.	$.089 l. = 1 s. 9 d. 2 f.$
2.	$.35 l. = 7 s.$	6.	$.67 l. = 13 s. 5 d.$
3.	$.248 l. = 4 s. 11 d. 3 f.$	7.	$.038 l. = 9 d. 1 f.$
4.	$.317 l. = 6 s. 4 d.$	8.	$.04 l. = 9 d. 3 f.$

If you compare thefe *Examples* with the *Rule*, the manner of finding the Value will be plain, without any further Explication.

The *Reafon* of this Rule is thus:

1. Since 1 Shilling is the $\frac{1}{20}$ Part of a Pound, and double any Number of 10th Parts, makes fo many 20th Parts; (fo $\frac{1}{10} = \frac{2}{20}$) therefore double the Figure in the 1/t Place (whofe Den^r is 10th Parts) is equal to fo many 20th Parts, or Shillings. Again,

2. Since $\frac{1}{20} = \frac{5}{100}$, therefore 5 in the 2^d Place (whofe Den^r is 100th Parts) is 1 Shilling. Then,

3. The Figure in the 3^d Place has 1000th Parts for its Den^r, and this with the Number over 5 in the 2^d Place, makes fo many 1000th Parts; which is little lefs than fo many Farthings; becaufe 1 Farthing is $\frac{1}{960}$ Part of a Pound. But when we make up a Decimal Table for Farthings from 1 to 47, (which is 11 d. 3 f.) we find this true in Fact, That from 1 to 23 Farthings the Figures in the 2^d and 3^d Places of the Decimal are the fame with the Number of Farthings: But from 24 to 47, the Figures in the 2^d and 3^d Places make a Number 1 more than the Number of Farthings. And tho' in all thefe Decimals (except that for 6 d. or 24 f.) there are Figures after the 3^d Place, yet their Value is not 1 Farthing, becaufe they do not make .001, which is lefs than 1 Farthing.

II. *To find the Decimal of a Pound, anfwering to any Number of Shillings, Pence, and Farthings, lefs than a Pound, without Tables or Pen.*

Rule 1. If the Number of Shillings is even, take its Half and fet in the firft Place after the Point, (*Ex.* 1.) If it's odd, fet the Half of the next leffer even Number in the firft Place, and 5 in the 2^d, (*Ex.* 2.) then reduce the d. and f. to f. and if they are fewer than 24 (i. e. the d. fewer than 6 = 24 f.) fet that Number in the 2^d and 3^d Places (i. e. in the 3^d Place if it's but one Figure, (*Ex.* 3.) and if it has two, add that which is in the Place of Tens to the Figure ftanding already in the 2^d Place, if there is any, and fet the other in the 3^d Place, *Ex.* 4.) But if thefe Farthings exceed 23 (i. e. if the d. exceed 5) add 1 to them, and fet that Number in the 2^d and 3^d Places as before, (*Ex.* 5, 6.) Thus you have the Decimal fought, true to three Places, which is fufficient for common ufe. But,

2. If you would compleat the Decimal, then if the Number of f. to which the d. and f. in the Queftion are equal, do not exceed 23, take that Number of f. or, if they exceed 23, take the Remainder after 24 is fubtracted from them, and divide that Number or Remainder decimally, (viz. by prefixing o's to it) by 24 (which is eafily and readily done by 4 and 6) the Quote, which will either be determinate, or circulate on 6 or 3, being fet after the Figures already found, the Decimal is compleated.

Examp. 1.	$4 s. = .8 l.$	*Examp.* 6.	$9 s. 9 d. = .4875$
2.	$13 s. = .65$	7.	$7 d. 3 f. = .0322916$ &c.
3.	$6 s. 2 f. = .308$	8.	$4 s. 8 d. 3 f. = .2364583$ &c.
4.	$8 s. 3 d. = .4125$	9.	$5 s. 4 d. 1 f. = .2677083$ &c.
5.	$7 s. 6 d. = .375$		

The

The *Reason* of this *Rule* is this:

1. For the Shillings: The half of any Number of 20th Parts (*i. e.* of any Number of Shillings) makes so many 10th Parts; and if there is an odd Shilling, it is equal to 5 in the 2^d Place, or $\frac{5}{100}$ Parts, because $\frac{5}{100}$ is $\frac{1}{20}$.

2. For the Pence and Farthings: If we make a Table of Decimals for any Number of Farthings from 1 to 47 (equal to 11 *d.* 3 *f.*) then for any Number less that 24 (or 6 *d.*) the Decimal has that Number in the 3^d, or 2^d and 3^d Places, (*i. e.* it will have so many 1000th Parts. And if these Farthings exceed 23, the Decimal has 1 more 1000th Parts. Again, For any Number of Farthings less than 24 (or 6 *d.*) consider, that because 1 Farthing is $\frac{1}{960}$ Part of a Pound, which is greater than $\frac{1}{1000}$ Part; therefore besides so many 1000th Parts, there must be added such a Decimal as is equal to the Difference of so many 960th Parts and 1000th Parts. Now, if we subtract $\frac{1}{1000}$ from $\frac{1}{960}$ the Remainder is $\frac{1}{24000}$; wherefore any Number less than 24, of 24000th Parts, will make a Decimal, whose first Place will fall in the fourth Place after the Point : Consequently this Decimal which remains to compleat the Decimal sought, falls after the 3 Places already found.

Lastly, If the Number of Farthings exceeds 23, it is either 24 ; and then there is nothing to be added to the Number of 1000 th Parts already set down : Or it's more than 24. And what's more, being less than 23 ; the Decimal to be added for that, comes under the same Rule as the last Article.

QUESTIONS, *shewing the* Application *of* Decimals *in Multiplication and Division.*

Quest. 1.   There is 14 *l* : 8 *s* : 6 *d* in each of 6 Bags: How much is in the whole? *Answer*, 86 *l* : 11 *s*.   Thus, 14 *l* : 8 *s* : 6 *d* is 14.425 *l.* which multiplied by 6 produceth 86.55 equal to 86 *l* : 11 *s*.

Quest. 2.   If 1 Yard of Cloth cost 14 *s* : 8 *d.* what is the Value of 24 *yds* : 3 *qr* : 1 *na* *Anf.* 18 *l* : 3 *s* : 2 *f.* nearest.   Thus, 14 *s* : 8 *d* is .7333 *&c. l.* and 24 *yds* : 3 *qr* : 1 *na* is 24.7625 *yds*: Which multiplied by .7333 produces 18.158 *&c. l.* which is 18 *l* : 3 *s* : 2 *f* nearly.

Quest. 3.   If 30 *l.* buy 124 *yds* : 1 *qr* : 2 *na.* of Cloth, How much will 1 *l.* buy? *Anfw.* 4 *yds* : 2 *na.* and .32 nearly.   Thus, 124 *yds* : 1 *qr* : 2 *na.* is 124.375, which divided by 30, quotes 4.145 *&c.* which is 4 *yds* : 2 *na.* and .32 nearly.

Quest. 4.   If 8 *l* : 9 *fb* : 4 *d.* buy 3 hundred Weight : 1 *qr* : and 18 Pound of Sugar, What may be bought for 1 *l*? *Anfw.* 1 *qr* : 17 *lb* and .1 nearly.   Thus, 8 *l* : 9 *fb* : 4 *d.* is 8.4666 *&c.* and 3 *C* : 1 *qr.* 18 *lb*. is 3.410714, which divided by the other, quotes .4028 *&c.* which is 1 *qr* : 17 *lb* and .1 nearly.

Observe, The Use and Application of Decimals will more fully appear in the remaining Parts of this Work ; especially by applying them in *Book 6*.   As to which Application, this in general only needs to be further said here, That any Integer being consider'd as the highest Denomination, all Numbers or Quantitys less than that are to be expressed decimally by taking the Decimal of that Integer answering to that lesser Quantity ; and in the same Question using Decimals of the same Integer for all Numbers of the same kind, (*i. e.* for all Numbers of Money, use the same Integer as 1 *l.*)   Then multiply and divide by these Numbers according to the Rules of Decimals.   And in Multiplication you may use the manner of Contraction explained in *Ch.* 8. §. 5.

I must *observe* in the last place, that most Questions in common Business are sooner done without Decimals, by the common Methods of Reduction ; but when to use Decimals, or the common Methods, must be left to every body's own choice : and indeed a good deal of Practice will be necessary to enable one to chuse judiciously.   And particularly as to the preceding Examples, *observe*, That the first is easier done by the common Method ; because it can be done without Reduction, the Multiplier being small. The second and fourth cannot be done by any Method hitherto explain'd, (except by Decimals ;) because by the Method of Reduction they require both Multiplication and Division ; as you will afterwards understand in the *Rule of Three*, (*Book 6*.)   The third may be easily done, the common way.   B O O K

BOOK III.

Of the Powers and Roots of Numbers.

CHAP. I.

Containing the THEORY.

DEFINITIONS.

I. A *Power* is a Number, which is the Product of a certain Number of equal Factors, *i. e.* of the fame Number multiply'd into itfelf continually a certain Number of Times.

II. A *Root* is a Number, by whofe continual Multiplication into itfelf another Number (which is called the Power) is produced.

Example. Let any Number 2 be multiplied into itfelf, the Products are 4 (= 2 × 2) 8 (= 4 × 2) 16 (= 8 × 2) &c. Then is 2 called the Root of thefe Products, which are called the Powers of that Root.

Hence it is plain, that Power and Root are relative things. Every Power is the Power of fome Root, and every Root is the Root of fome Power: So that by calling one Number the Power or Root of another, we mean that it is the Number produced, or the Number producing that other by continual Multiplication.

But the feveral Powers, and the Root in relation to thefe, are alfo diftinguifhed by particular Names, which fhall next be explained.

III. The firft Product (*viz.* that of the Root multiplied by itfelf) is called the Square of the Number multiplied; which in refpect of the other is called the Square Root. So 4 is the Square of 2, and 2 the Square Root of 4, becaufe 2 × 2 = 4.

The fecond Product (*viz.* of the Square by the Root) is called the Cube, and the Root in refpect of it is called the Cube Root: So 8 is the Cube of 2, and 2 is the Cube Root of 8; for 2 × 2 × 2, or 4 × 2 = 8.

Others of the Powers had alfo particular Names among the Ancients; but they are very complex and burthenfome to the Memory, and tend no way to the Improvement or Eafinefs of the Science: Whereas it is obvious that we have no more to do, but diftinguifh them by their Order in the Series of Products, calling the firft Product the firft Power, the fecond Product the fecond Power, and fo on; whereby thefe Names do of themfelves in a very fimple and eafy manner diftinguifh the feveral Powers, in confequence of the general Definition of a Power: for they exprefs the Number of Multiplications of the

T.   Root

Root in the Production of each Power; which the ancient Names do not. For the Names Square and Cube, of which the reft were compounded, are Names of geometrical Quantities applied to Numbers, only from this Confideration, that the Meafures of thefe Quantities are found by fuch an Application of Numbers, as do produce the Numbers, which are hence called *Square* and *Cube*.

But obferve again, that tho' in confequence of the preceding Definitions of Power and Root, thefe Terms ought always to be contradiftinguifhed, fo that the Products only can be called Powers; yet for the fake of a particular Conveniency, which we fhall prefently underftand, the Root is called the firft Power, and the Products in order are called the fecond, third, &c. Power, as here:

Root or firft Power.	Square or 2d Power.	Cube or 3d Power.	4th Power.	5th Power.
2.	2 × 2.	4 × 2.	8 × 2.	16 × 2. &c.
2.	4	8	16	32 &c.

In which Method the Root is the fame with the firft Power, and contradiftinguifhed only from the fuperior Powers, with refpect to which we call it the fecond or third, &c. Root; tho' more commonly we ufe the Names Square and Cube, and Square Root, Cube Root; ufing the Names fourth, fifth, &c. Power and Root, for the degrees above the Cube or third Degree.

Of the univerfal Notation of Powers and Roots.

I. Of POWERS.

TAKE any Number A for a Root, and the Series of its Powers according to the Definition will be thus:

Root, or 1ft Power.	2d Power.	3d Power.	4th Power.
A.	AA.	AAA.	AAAA. &c.

Each of thefe Terms expreffing the continual Product of A, taken fo oft as it is placed in each of them, which being once more at every Step gradually from the Root, we have alfo this more convenient Method of expreffing them, *viz.* by writing only the Root with a Number over it, to fignify how oft the Root is to be taken, or placed as a Factor in producing that Power. Thus the *4th* Power of A is AAAA, to be written, according to this other Method, thus, A^4; and fo of others, the whole Series of the Powers being reprefented thus:

$$A^1, \; A^2, \; A^3, \; A^4, \; A^5, \; A^6, \; \&c.$$

When a Number A has no Figure or Mark of Power, it's fuppofed to be the firft, fo that A^1 or A are equivalent.

Thefe Figures we call *Indexes* or *Exponents* of the Powers; becaufe by fhewing the Number of Factors, they fhew what Power is fignified by that Expreffion, or what Term in order of the Series; for the Numbers of Factors increafe gradually in the Series, the Root

Root ftanding alone in the 1*ft* Term, twice in the 2*d*, and fo on. And fince the Denominations of the Powers are taken from their Places in the Series, they do alfo exprefs the Number of equal Factors, or the Number of times the Root is placed by Multiplication in every Power, and confequently the Index is the Denomination; fo if $A = 2$ then $A^3 = AAA = 2 \times 2 \times 2 = 8$.

Again: By this Method any Power indefinitely may be expreffed by a general or indefinite Index thus, A^n; which is any Power of A, according to the Value we put upon the Index n.

Hence any Series of Powers decreafing from a given one down to the Root may be expreffed thus:

$$A^n, A^{n-1}, A^{n-2}, A^{n-3}, A^{n-4}, \&c.$$

Still fubftracting one more from the Index till it become equal to 1, and then you have the Root.

II. *For* ROOTS.

The Root of any Number confidered as a Power may alfo be conveniently expreffed by that Number with an Index; thus, over the Number which is the Denomination of the Root, fet 1, in form of a Fraction; this is the Index of the Root: For Example; The Square Root of A is $A^{\frac{1}{2}}$, the Cube Root $A^{\frac{1}{3}}$, the 4th Root $A^{\frac{1}{4}}$, and fo on; fo that if $A = 4$, then $A^{\frac{1}{2}} = 2$: Or if $A = 8$, then $A^{\frac{1}{3}} = 2$; And an indefinite Root thus, $A^{\frac{1}{n}}$.

There is alfo another way of marking Roots by this Mark $\sqrt{}$, fetting the Power before it, and the Index above it: Thus the Square Root of A is $\sqrt[2]{A}$, the n Root is $\sqrt[n]{A}$.

And now, to underftand the Conveniency of diftinguifhing the Powers by their Order in the Series, *i. e.* by the Number of Factors or Indexes, Confider that the various Powers of the fame Root differ only by thefe Indexes, or Numbers of Factors; and the Rules for their mutual Application to one another by Multiplication and Divifion, (by which chiefly their different Properties are difcovered,) depending upon the Confideration of thefe different Numbers of Factors, it is a more fimple and eafy Method to make the fame Number exprefs both the Number of Factors, and give a Denomination to the Power; which would not be, if we fhould begin the Numeration of the Powers at the firft Product, calling AA, or A^2 the Firft Power. It is true indeed, that by this Method the Denomination would always be one lefs than the Index or Number of Factors, and fo would be a certain regular Method of fhewing that Number; but ftill the other is more fimple and eafy: Which the Applications to be made afterwards will make appear more evidently.

There is one thing more you may obferve upon this Method of denominating Powers, *viz.* That tho' the Root is not a Product of itfelf multiplied into itfelf, and fo is not a Power according to the general Definition; yet we may always contra-diftinguifh Root and Power, underftanding them according to the general Definition, and at the fame time take the Denominations of Powers from the Indexes or Numbers of Factors; provided we underftand thefe Denominations or Indexes to exprefs no other thing but the Number of Factors, *i. e.* a Power compofed of fo many Factors as the Index expreffes, and not as fignifying the Degree and Order of the feveral Powers from the firft Product, which, according to the general Definition, is the firft Power, tho' the Index is 2; fo for Example, A^4 is called the Fourth Power, not as being the fourth Term in the order of Products, for it is only the 3d Product, but as being compofed of four Factors; *viz.* the Root ftated as a Factor four times; fo $A^4 = A \times A \times A \times A$.

But now after all, it's to the fame Purpofe in which of thefe Views you take the Denomination; for the whole Conveniency lies in having the Number of Factors expreffed, which is done either way. Others again confider 1 as a Factor in every Power, and then

they

they make the Index expreſs the Number of Multiplications by which a Power is produced: Thus $A^2 = 1 \times A \times A$; in which are two Multiplications, 1ſt $1 \times A$, 2d $1 \times A$ by A.

DEFIN. IV. *Powers* or *Roots* are called *Like* or *Similar* to one another, whoſe Denominations or Indexes are the ſame; ſo A^2, B^2, or A^n, B^n, are ſimilar Powers, and theſe ſimilar Roots, $A^{\frac{1}{n}}$, $B^{\frac{1}{n}}$. Such are alſo ſaid to be Powers or Roots of the ſame Degree or Order.

And when the Indexes are different or unequal, they are called *unlike* or of a different Order: As A^n, B^m; alſo $A^{\frac{1}{n}}$, $B^{\frac{1}{m}}$.

V. The finding any Power of a Number is called *Raiſing* that Number to ſuch a Power; as finding the 4th Power of A is called Raiſing A to the 4th Power: And this is alſo call'd in general, *Involving*, or the *Involution* of that Number, according to the Index of the Power.

VI. The finding any propoſed Root of a Number, is called the *Extracting* of ſuch a Root from that Number; as finding the Cube Root of A is call'd the extracting the Cube Root of A; and this we call in general *Evolving*, or the *Evolution* of that Number, according to the Index of the Root.

VII. As any Number may be made a Root, and involved to any Power, ſo if a Number C is a Power of another B, which is again a Power of another A, then may C be called a *Compound* Power of A, *i. e.* a Power of a Power of A, (as with reſpect to B it's a ſimple Power,) and may be generally expreſſed thus: $A^{\overline{m}|n}$, that is, the n Power of A^m. Example: 64 is the Square of the Cube of 2, for it is the Square of 8, which is the Cube of 2. The Compoſition may alſo conſiſt of more than two Members, as the n Power of the m Power of A^o.

VIII. If any Number A is a certain Root of another B, which is alſo a certain Root of another C, then may A be called a compound Root of C (as with reſpect to B it is a ſimple Root) and may be expreſſed thus, $C^{\overline{\frac{1}{m}}|\frac{1}{n}}$, that is, the n Root of $C^{\frac{1}{m}}$. Example: 2 is the Cube Root of the Square Root of 64.

SCHOLIUM. That theſe Compound Powers and Roots muſt be equal to ſome immediate or ſimple Power or Root of the Number to which they are referred, will eaſily be underſtood from the Nature of ſuch Numbers; that is, the n Power of the m Power of A is ſome immediate ſimple Power of A, as A^o; and ſo of Roots: How ſuch ſimple Expreſſions are found, ſhall be explained in its place.

IX. A Number which is firſt conſidered as a certain Root of another, as the n Root, may be itſelf involved according to ſome other Index m, and this Power being referred to the ſame Number to which the preceeding Root was referred, may be called a *mix'd* Power of that Number, ſo if $B = A^{\frac{1}{n}}$, then the m Power of B, that is, the m Power of the n Root of A is a mix'd Power of A (which referred to B is a ſimple Power) and may be expreſſed $A^{\frac{1}{n}m}$. Example: 9 is the Square of the Cube Root of 27, for it is the Square of 3, whoſe Cube is 27.

In the ſame manner, a Number being conſidered as a certain Root of a certain Power (whoſe Index is different from that of the Root) of a Number, it may be called a mix'd Root, as the m Root of the n Power of A, repreſented thus: $A^{\overline{n}|\frac{1}{m}}$. Example: 9 is the Cube Root of the Square of 27, for the Square of 27 is 729, which is alſo the Cube of 9.

of 9.   2   *Obſerv.*

Obferve, For either of thefe Kinds, *viz.* a mix'd Power, or mix'd Root, we may inftitute this manner of Reprefentation $A^{\frac{n}{m}}$, which may fignify either the m Root of A^n, or the n Power of $A^{\frac{1}{m}}$. But then obferve, that we can't make it reprefent either of thefe indifferently, till we have firft demonftrated that they are equal; which fhall be afterwards done; and till then, I fhall only ufe it for the m Root of the n Power.

S c h o l i u m. Every Number is a Root of any Order whatever, becaufe it may be *involved* to any *Power*; but every Number is not a Power of any Order; fome being Powers of no Order but the firft, which is only being a Root; *i. e.* there are fome Numbers which cannot be produced by the continual Multiplication of any Number whatever; and fuch are 3, 5, 6, 7, and an infinite Number of others. Some again are Powers of one particular Order only; as 4, which is only a Square; and 8, which is only a Cube. Some, in the laft place, are Powers of more than one Order, but limited to a certain Number of different Orders; as 64 is both a Square and a Cube; its Square Root being 8, and the Cube Root 4; for $8 \times 8 = 4 \times 4 \times 4 = 64$: The Demonftration of thefe things you'll learn afterwards; to mention them in general is enough here, which was only neceffary for the fake of the following Definition.

D e f i n. X. When a Number A is propofed as a Power of any Order n, and yet is not a Power of that Order, *i. e.* if it has no determinate Root of that Order, or there is no Number which involved as the Index n directs, will produce that Number; yet it has what we may call an indeterminate Root, (as fhall be afterwards explain'd) and this imagin'd Root, under the Notion of a true and compleat Root, is called a *Surd* (*i. e.* inexpreffible) Root, and is reprefented in the general Form $A^{\frac{1}{n}}$; and fuch Roots as are real, are, in Diftinction from Surds, called *Rational Roots*. *Exam.* 8 is not a Square; for there is no Number, which multiplied into itfelf, will produce 8. No Integer will, fince $2 \times 2 = 4$, and $3 \times 3 = 9$; and that no mix'd Number betwixt 2 and 3 can do it, will be afterwards demonftrated.

But now as to Surds, don't miftake, as if fuch Roots or Reprefentations were nothing at all, or fo merely imaginary as to be of no Ufe in Arithmetick; for though there be no fuch determinate or affignable Number, whofe n Power is equal to A; yet we can find Numbers mix'd of Integers and Fractions, that fhall approach nearer and nearer to the Condition required, *in infinitum*; *i. e.* we can find a Series of Numbers decreafing continually, whofe Sum taken at every Step is a Number, the Power of which approaches nearer and nearer to the given Number; and this Series confider'd in its infinite Nature, as going on by the fame Tenor and Law without end, and thereby approaching infinitely near to the Condition of a true Root, is truly and properly what we call a Surd; which, 'tis plain, is fomething real in itfelf, though we can't exprefs the whole Value of it by any definite Number; for that is contrary to its Nature: So we find that the Surd Roots of different Numbers have certain Connections and relative Properties the fame way as rational Numbers have; (all which things fhall be demonftrated in their proper place.)

Therefore we conceive Surds as Quantities compleat of their own kind, and fo ufe the fame general Notation for Surds and rational Roots: And hence the following Theory relating to Roots are to be underftood generally, whether they are Surds, or rational Roots; concerning the Reafon and Application of which to Surds, you'll learn more Particulars afterwards in *Chap.* 3.

XI. The *Powers* and *Roots* of *Fractions* are to be underftood the fame way as of whole Numbers; that is, any Fraction being continually multiplied into itfelf, is a *Root* or *firft Power*, with refpect to the Products which are the fuperior Powers.

Exam-

Example. $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and $\frac{4}{9}$ is therefore the Square of $\frac{2}{3}$, and may be univerfally expreffed thus, $\overline{\dfrac{a}{t}\Big|}^{n}$; or thus, $\dfrac{a^{n}}{b^{n}}$; and the Root thus, $\overline{\dfrac{a}{b}\Big|}^{\frac{1}{n}}$; or thus $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$.

Obferve, That only is the proper and immediate Power of a Fraction, whofe Terms are the Powers of the Terms of that Fraction; yet as the fame Fraction may be exprefs'd in various Terms, fo all equivalent Fractions may be taken for the Power or Root of the fame Fraction, becaufe they have the fame Effect in all Operations, if any one of them is fo, according to the Definition. Thus, becaufe $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and becaufe $\frac{4}{6} = \frac{2}{3}$, therefore $\frac{4}{9}$, which is the Square of $\frac{2}{3}$, may be alfo called the Square of $\frac{4}{6}$, alfo $\frac{8}{18}$ ($=\frac{4}{9}$) may be called the Square of $\frac{2}{3}$ or $\frac{4}{6}$.

SCHOLIUM. When a complex literal Expreffion is confider'd as a Root, to exprefs any Power of it we draw a Line over the whole Expreffion, and then annex the Index; thus, $\overline{A+B}^{n}$ is the n Power of $\overline{A+B}$, and $\overline{AB}^{n}$ is the n Power of the Product $A \times B$: But if there is no Line over, then the Index is applied only to the Member over which it is immediately fet, as $A + B^{n}$ is only the Sum of A and B^{n}, and $A B^{n}$ the Product of $A \times B^{n}$.

AXIOMS.

Ift, LIKE *Powers* or *Roots of equal Numbers* (however differently expreffed) are equal. Thus: If $A = B$, then $A^{n} = B^{n}$, and $A^{\frac{1}{n}} = B^{\frac{1}{n}}$. But unequal Numbers have their fimilar Powers and Roots unequal; the greater having the greater Number for its Power or Root. Hence,

COROL. If the different Powers of unequal Numbers are equal, the Power of the leffer Root has the greater Index. Thus: If $A^{n} = B^{m}$, and A be lefs than B; then is n greater than m: For if $n = m$, then is A^{n} lefs than B^{m}; and much more is it fo, if n is lefs than m. *Exam.* $8^{2} = 4^{3} = 64$.

IId, If a Number is involved to any Power, and from this Power a Root of the fame Denomination is extracted, this Root is the fame Number which was firft involved. So if A is involved to the n Power, and of this Power, *viz.* A^{n}, we extract the n Root, it is equal to A, *i. e.* $A = \overline{A}^{\frac{1}{n}}$. And reverfly extract any Root of a Number, and then involve that Root to a Power of the fame Denomination; this Power is the fame Number from which the Root was firft extracted; fo the n Power of $A^{\frac{1}{n}}$, or $\overline{A^{\frac{1}{n}}}^{n} = A$. Hence,

COROL. 1. An Expreffion of this Sort $A^{\frac{n}{n}}$, where the Numerator and Denominator of the Index are equal, whether it is underftood as the n Root of the n Power; or the Power of the n Root of A, is no other thing in Effect and Value but A. Hence again;

2. Involution and Evolution are directly oppofite, the one undoing the Effect of the other; whereby they are mutual Proofs one of the other.

THEOREM I.

IF two fimilar Powers of different Numbers or Roots are multiplied together, the Product is the like Power of the Product of thefe Numbers or Roots. Thus, the Product of two Squares is the Square of the Product of their Roots. *Univerfally* $A^{n} \times B^{n} = \overline{AB}^{n}$.

Demonftr

$A = 2, \; A^2 = 4$
$B = 5, \; B^2 = 25$
$\overline{AB} = 10. \; A^2 \times B^2 =$
$100 = \overline{AB}^2.$

Demonſtr. In the Product $A^n \times B^n$, A and B are applied as Factors equal times; fo that in the whole Product there is a Number of Factors equal to $2n$; and confequently the Product A B, confider'd as one Number or Factor, is applied in that Product n times [for in whatever Order the Factors are taken, the Product is ſtill the fame]; fo that the Product given is A B raifed to the n Power, or $\overline{AB}^n$. Thus particularly, $A^3 \times B^3 = AAA \times BBB = AB \times AB \times AB = \overline{AB}^3$.

Co rol. Hence we learn, how the Product of a Number expreſſed as a Power, and another not expreſſed as fuch (but ſuppofed to be one) may be reduced to fuch an Expreſſion. Thus, $A^n \times B = \overline{A \times B^{\frac{1}{n}}}^n$; for the n Power of A is A^n, and the n Power of $B^{\frac{1}{n}}$ is B, therefore $A^n \times B = \overline{A \times B^{\frac{1}{n}}}^n$.

Schol. This Theorem is true alfo, when the Roots are the fame; as $A^n \times A^n = \overline{AA}^n$. But I have not taken this Cafe into the *Theorem*, becaufe it falls within another (fee *Theor.* 6.) where the Product is expreſſed in a more fimple and convenient way. You are to underftand the fame of the *Theorems* 2, 3, and 4.

THEOREM II.

IF two fimilar Powers of different Numbers or Roots are divided one by the other, the Quote is the like Power of the Quote of the given Roots. Thus the Quote of two Squares is the Square of the Quote of their Roots *univerfally* $B^n \div A^n = \overline{B \div A}^n$.

$A = 3. \; A^3 = 27$
$B = 6 \; B^3 = 216$
$B \div A = 2. \; B^3 \div A^3 = 8$
$\overline{B \div A}^3 = 8.$

Demonſtr. 1. Suppofe A, B are Integers, then $B \div A$ expreſſed fractionally is $\dfrac{B}{A}$, whofe n Power, according to *Defin.* 11. is $\dfrac{B^n}{A^n} = B^n \div A^n.$

2. Suppofe B and A fractional; thus, $B = \dfrac{c}{o}$ and $A = \dfrac{d}{a}$, then is $B^n = \dfrac{c^n}{o^n}$ and $A^n = \dfrac{d^n}{a^n}$.

Alfo $B^n \div A^n = \dfrac{c^n}{o^n} \div \dfrac{d^n}{a^n} = \dfrac{c^n a^n}{o^n d^n} = \dfrac{\overline{c a}^n}{\overline{o d}^n}$ *(Theor.* 1.) But $\dfrac{ac}{od} = \dfrac{c}{o} \div \dfrac{d}{a} = B \div A$, and its n

Power is $\dfrac{\overline{a c}^n}{\overline{o d}^n}$, which is $\overline{B \div A}^n$.

Scholium. I have not here confidered, whether $B^n \div A^n$ is a whole Number or a Fraction; for which fo ever of them it be, you fee plainly that it is equal to $\overline{B \div A}^n$. In another place you'll find it demonftrated, that according as $B^n \div A^n$ is integral or fractional, fo is $B \div A$; and reverfly.

The following Corollaries 2, 3, 4, *and* 5, *are deduced from this and the firſt* Theorem *jointly confidered.*

Coroll. 1. The Quote of two Numbers, whereof one is expreſſed as a Power and the other not, (tho' ſuppofed to be one,) may be reduced to fuch an Expreſſion thus; $A^n \div B = \overline{A \div B^{\frac{1}{n}}}^n$; for A^n is the n Power of A, and B is the n Power of $B^{\frac{1}{n}}$.

2. If any Product and one of the Factors are fimilar Powers, the other is alfo a fimilar Power, and its Root is the Quote of the Roots of the other two Terms. Thus, if
A^n

$A^n \times B = D^n$, then is $B = D^n \div A^n = \overline{D \div A}\,^n$ whose n Root is $D \div A$. In another Place it shall be demonstrated that $D \div A$ must be integral if $D^n \div A^n$ or B is so.

III. If the Dividend and either of the other two Terms, *viz.* Divisor and Quote, are similar Powers, the other of those Terms is also a similar Power, whose Root is the Product of the Roots of the Dividend and the former of the other two Terms. Thus if $B^n \div A = D^n$, then $A = B^n \times D^n = \overline{BD}\,^n$.

4. The Product of two Numbers being a Power, the Factors are either both like Powers with the Product, or neither of them is so. Also, if one Factor is a Power, and the other not a like Power, the Product is not a like Power. Let $AB = p^n$, if A is a Power of the Order n, so is B, for if $A = a^n$ then is $a^n \times B = p^n$, and consequently B is of the Order n (by the 2d) whence also if the one A or B is not a Power of the Order n, neither is the other; for one being so, the other would be so too. Again, if $A^n \times B = D$, and B not a Power of the Order n, neither is D; for if D were so, B would be so too.

5. If the Quote of two Numbers is a Power, these Numbers are either both Powers like the Quote, or neither of them is so: This is the Reverse of the last.

SCHOLIUM. In these Corollaries you are to understand Rational Powers; for otherwise any Number may be represented as a Power of any Order.

THEOREM III.

If two similar Roots of different Numbers are multiplied, the Product is the like Root of the Product of these Numbers. Thus two Cube Roots produce a Number which is the Cube Root of the Product of their Cubes. Universally, $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{AB}^{\frac{1}{n}}$.

Example:

$A = 27. \quad A^{\frac{1}{3}} = 3$

$B = 8. \quad B^{\frac{1}{3}} = 2$

$AB = 216. \quad A^{\frac{1}{3}} \times B^{\frac{1}{3}} = 6$

$\overline{AB}^{\frac{1}{3}} = 6$

Demonstration This is but the Reverse of the 1st *Theorem*, and follows easily from it. Thus, Let A B be any similar Powers, then is $A \times B = \overline{A^{\frac{1}{n}} \times B^{\frac{1}{n}}}^{n}$ by *Theor.* I. and by *Ax.* 1. $\overline{A \times B}^{\frac{1}{n}} = A^{\frac{1}{n}} \times B^{\frac{1}{n}}$, which is the n Root of $\overline{A^{\frac{1}{n}} \times b^{\frac{1}{n}}}^{n}$.

COROLL. The Product of two Numbers, whereof one is expressed as a Root and the other not, may be reduced to such an Expression thus; $A^{\frac{1}{n}} \times B = \overline{A \times B^n}^{\frac{1}{n}}$; for A is the n Power of $A^{\frac{1}{n}}$, and B^n of B; therefore, by this Theorem $A^{\frac{1}{n}} \times B = \overline{A \times B^n}^{\frac{1}{n}}$. This is also the Reverse of *Coroll. Theor.* I.

THEOREM IV.

If two similar Roots of different Numbers are divided, one by the other, the Quote is the like Root of the Quote of the one Number divided by the other. Thus, two Cube Roots give for a Quote the Cube Root of the Quote of the Cubes. Universally, $D^{\frac{1}{n}} \div A^{\frac{1}{n}} = \overline{D \div A}^{\frac{1}{n}}$.

Example:

$D = 216. \quad D^{\frac{1}{3}} = 6$

$A = 27. \quad A^{\frac{1}{3}} = 3$

$D \div A = 8 \quad D^{\frac{1}{3}} \div A^{\frac{1}{3}} = 2$

$\overline{D \div A}^{\frac{1}{3}} = 2$

Demonstr. 1. Suppose $D^{\frac{1}{n}}, A^{\frac{1}{n}}$ are both Integers, then are D, A also Integers, (*viz.* the Powers, or Products of integral Factors,) therefore $D^{\frac{1}{n}} \div A^{\frac{1}{n}}$ or $\dfrac{D^{\frac{1}{n}}}{A^{\frac{1}{n}}}$ is a Fraction in Terms

whose n Power is $\dfrac{D}{A}$; *i. e.* the n Root of $\dfrac{D}{A}$ is $\dfrac{D^{\frac{1}{n}}}{A^{\frac{1}{n}}}$ or $D^{\frac{1}{n}} \div A^{\frac{1}{n}}$.

2. If

2. If $D^{\frac{1}{n}}, A^{\frac{1}{n}}$ are fractional, then are D, A also fractional, (according to a former Observation.) Suppose $D^{\frac{1}{n}} = \frac{b}{a}$, and $A^{\frac{1}{n}} = \frac{d}{c}$, then is $D = \frac{b^n}{a^n}$, and $A = \frac{d^n}{c^n}$ (*Ax.* 1.) Also

$$D^{\frac{1}{n}} \div A^{\frac{1}{n}} = \frac{b}{a} \div \frac{d}{c} = \frac{bc}{ad}; \text{ and } D \div A = \frac{b^n}{a^n} \div \frac{d^n}{c^n} = \frac{b^n c^n}{a^n d^n} = \overline{\frac{bc}{ad}}^n; \text{ therefore } \overline{D \div A}^{\frac{1}{n}} =$$

$$\frac{bc}{ad} = D^{\frac{1}{n}} \div A^{\frac{1}{n}}.$$

C o r o l l. The Quote of two Numbers whereof one is expressed as a Root, and the other not, may be reduced to such, thus; $D^{\frac{1}{n}} \div A = \overline{D \div A^n}^{\frac{1}{n}}$. For $D^{\frac{1}{n}}$ is the n Root of D, as A is of A^n.

S c h o l i u m. From the two last Theorems jointly consider'd, we have four *Corollaries* after the same manner as the 2*d*, 3*d*, 4*th*, 5*th* Corollaries, deduced from *Theorem* II*d* and I*ft*.

T h e o r e m V.

T h e Product or Quote of any like Mixt Power (or Root) of two different Numbers, is the like Mixt Power of the Product or Quote of these two Numbers: Thus

$$A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}} \text{ and } A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}}.$$

Demonstr. 1. $A^n \times B^n = \overline{AB}^n$ (*Theor.* I.) and $\overline{A^n}^{\frac{1}{m}} \times \overline{B^n}^{\frac{1}{m}} = \overline{A^n \times B^n}^{\frac{1}{m}}$ (*Theor.* III.) But $\overline{A^n}^{\frac{1}{m}} = A^{\frac{n}{m}}$ and $\overline{B^n}^{\frac{1}{m}} = B^{\frac{n}{m}}$ by the Notation; and since $A^n \times B^n = \overline{AB}^n$, hence $A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}}$.

2. $A^n \div B^n = \overline{A \div B}^n$ (*Theor.* II.) And $\overline{A^n}^{\frac{1}{m}} \div \overline{B^n}^{\frac{1}{m}} = \overline{A^n \div B^n}^{\frac{1}{m}}$, (*Theor.* IV.) $= \overline{A \div B}^{\frac{n}{m}}$, (*Ax.* I.) That is, $A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}}$.

Example. The Square Root of 16 is 4, and the Cube of 4 is 64, therefore the Cube of the Square Root of 16, or $16^{\frac{3}{2}}$ is $= 64$. In like manner, the Cube of the Square Root of 81, or $81^{\frac{3}{2}}$ is $= 729$. Then $81 \times 16 = 1296$, whose Square Root is 36, and the Cube of this is $46656 = 64 \times 729$; that is, $16^{\frac{3}{2}} \times 81^{\frac{3}{2}} = \overline{16 \times 81}^{\frac{3}{2}}$.

T h e o r e m VI.

I f two Powers of the same Root are multiplied, the Product is such a Power of the same Root, whose Index is the Sum of the Indexes of the Factors. Thus, The Product of the 2*d* and 3*d* Powers of any Number, is the 5*th* Power of the same Number. Universally, $A^n \times A^m = A^{n+m}$.

Example.

$A = 3.$ $A^2 = 9$
$A^3 = 27.$ $A^2 \times A^3 = 243$
$A^5 = 243.$

Demonstr. A^n and A^m being each a Product of A continually multiplied by itself, their Product must be a Product of A continually by it self; *i. e.* a Power of A. Also

U

Alfo it's plain, that in $A^n \times A^m$, the Root A is applied as a Factor as oft as the Sum $n + m$, fo that $A^n \times A^m = A^{n+m}$. Particularly, fuppofe $A^2 \times A^3$, the Product is $A A \times A A A = A A A A A = A^5$.

SCHOLIUM. What's here proved for two Factors, holds equally for three or more: Thus $A^n \times A^m \times A^r = A^{n+m+r}$.

COROLL. Hence we learn how to find any Power of a given Root, without finding all the intermediate Powers; *viz.* by multiplying together two or more Powers of that Root, the Sum of whofe Indexes is the Index of the Power fought: Thus, having the $3d$ and $4th$ Powers, their Product is the $7th$ Power.

THEOREM VII.

IF one Power is divided by another Power of the fame Root; the Quote is equal either to another Power of the fame Root, or to a fractional Power whofe Numerator is 1, and the Denominator fome Power of the fame Root. Thus, particularly, If the Dividend is greater than the Divifor, the Quote is a Power of the fame Root whofe Index is the Difference of the Indexes of the propofed Powers; and if the Dividend is lefs, the Quote is a fractional Power whofe Numerator is 1, and the Denominator is fuch a Power of the given Root whofe Index is the Difference of the given Indexes.

Thus $A^n \div A^m$ is either A^{n-m}, or $\dfrac{1}{A^{m-n}}$.

Example 1.

$A = 3 \quad A^4 = 81$

$A^2 = 9. \quad A^4 \div A^2 = 9$

$A^{4-2} = A^2 = 9$

Example 2.

$A^2 = 9 \quad A^5 = 243$

$A^2 \div A^5 = 9 \div 243 =$

$\dfrac{1}{A^{5-2}} = \dfrac{1}{A^3} = \dfrac{1}{81}$

Demonftr. This is the Reverfe of the laft *Theorem*, and the Reafon of it contain'd in that, from the reciprocal Nature of Multiplication and Divifion, with that of Addition and Subftraction. Or, this fubftracting of the one Index from the other, is only the taking out equal Factors from the Divifor and Dividend, *i.e.* dividing them equally; which makes the Quotes ftill the fame: Thus, $A^5 \div A^2 = A^3 \div 1 = A^3$, and $A^2 \div A^5 = 1 \div A^3$.

THEOREM VIII.

EVERY *Compound* Power (or Power of a Power) of any Root is equal to fuch a Simple Power of the fame Root whofe Index is the Product of the given Indexes: Thus the $3d$ Power of the $2d$ Power is the $6th$ Power. Univerfally, the n Power of the m Power of A, or $\overline{A^m}^n$ is $= A^{nm}$.

Example.

$A = 2. \quad A^2 = 4$

$A^6 = 64 = \overline{A^2}^3$

Demonftr. $A^m \times A^m = A^{2m}$ (*Theor.* 6.) and involving or multiplying A^m by itfelf once more, the Index of the Product contains m once more; fo that however oft A^m is employed as a Factor, as n times, the Index of the Product will be fo many times m, or $m \times n$: But A^m employed as a Factor n times makes the n Power of A^m; which is therefore equal to the mn Power of A, or A^{mn}.

SCHOLIUMS.

1. The fame Reafoning is good, when the Compofition confifts of more than two Steps: Thus the n Power of the m Power of the r Power, is the mnr Power of A, or A^{nmr}.

2. If the Index of any Power is the Product of two Numbers, it may be confidered as a Compound Power; or if it is the Product of more than two Factors, it may be reduced to a Compound of two, by taking any one of thefe Factors for one Member, and the Product of all the reft for the other. Thus, A^{nmr} is the r Power of A^{nm}, or the n Power of A^{mr}, or the m Power of A^{nr}. In fhort, if the Index of a Power is the Product

of

of other Numbers, whatever Variety is in the Compofition of that Product, there is the fame Variety in the Compofition of that Power.

COROLLARIES.

1. Here we learn to find a Number, which is a Power of as many different Orders as can be propofed, *viz.* by multiplying the Indexes of all thefe propofed Orders continually into one another, and raifing any Number to a Power, whofe Index is that Product. Thus: To find a Number which is both a Square and Cube, raife any Number to the fixth Power. Univerfally, to find a Number which is a Power of the Orders n, m, r, raife any Number A to the nmr Power, and A^{nmr} is the Number fought: for it is the r Power of A^{nm}, the n Power of A^{mr}, the m Power of A^{nr}.

Obferve alfo, That if one of the Indexes is an *aliquot* Part of another, we need not multiply that Index which is the *aliquot* Part. Thus: To find a Number which is both a fecond and fourth Power, we need only to take fome fourth Power, fince every fourth Power is alfo a fecond Power. For becaufe $2 \times 2 = 4$, therefore A^4 is the fecond Power of A^2. By this you'll underftand any other Cafe.

2. If out of a certain Power of a given Number, a Root is to be extracted of a different Index, and if the Name of the Root is an *aliquot* Part of the Name or Index of the Power; by dividing the Name of the Power by that of the Root, and applying the Quote as an Index to the given Number, we have an Exprefsion for the Root fought. Thus; the Square Root of A^6 is A^3, becaufe $6 \div 2 = 3$. And univerfally, let $n \div m = r$. Then is $A^{\frac{n}{m}} = A^r$, *i. e.* where there is a mixt or fractional Index (which exprefses a certain Root named by the Denominator, of a certain Power named by the Numerator) if the Denominator is an *aliquot* Part of the Numerator; then dividing the Numerator by the Denominator, the Quote is an Index, which applied to the fame Number, exprefses the Value of that mixt Root.

THEOREM IX.

EVERY Compound Root (or Root of a Root) of any Number, is equal to fuch a fimple Root of the fame Number, whofe Index is the Product of the propofed Indexes. Thus; the Square Root of the Cube Root is the fixth Root. Univerfally, the n Root of the m Root is the nm Root, or $\overline{A^{\frac{1}{m}}}^{\frac{1}{n}} = A^{\frac{1}{nm}}$.

Example:

$A = 64.$ $A^{\frac{1}{3}} = 4$

$A^{\frac{1}{2}} = 2 = \overline{A^{\frac{1}{3}}}^{\frac{1}{2}}$.

Demonftr. The Reafon of this is contained in the preceding, becaufe extracting of Roots is oppofite to raifing Powers. Or it may be demonftrated thus: Suppofe $A^{\frac{1}{nm}} = B$, then $A = B^{nm}$ (*Ax.* 1, 2. for A is the nm Power of its nm Root) and $A^{\frac{1}{m}} = B^{\frac{nm}{m}} = B^n$ (*Cor.* 2. *Theor.* 8.) Again, $\overline{A^{\frac{1}{m}}}^{\frac{1}{n}} = B$ (*Ax.* 1.) But $B = A^{\frac{1}{nm}}$; therefore $\overline{A^{\frac{1}{m}}}^{\frac{1}{n}} = A^{\frac{1}{nm}}$.

THEOREM X.

ANY Power of any Root of a Number is equal to the fame Root of the fame Power of that Number. Thus, the Square Root of the Cube of any Number is the Cube of the Square Root of that Number. Univerfally, $\overline{A^{\frac{1}{r}}}^n = \overline{A^n}^{\frac{1}{r}}$.

Example:

$A = 9.$ $A^{\frac{1}{2}} = 3$ $A^3 = 729.$

$A^{\frac{3}{2}} = 27 = \overline{A^{\frac{1}{2}}}^3$.

Demonftr. Suppofe $A^{\frac{1}{m}} = B$, then is $A = B^m$ (*Ax.* 1.) and $A^n = B^{mn}$ (*Ax.* 1. with *Theor.* 7.) Again, $\overline{A^n}^{\frac{1}{m}} = B^{\frac{mn}{m}} = B^n$ (*Ax.* 1.) $= B^n$ (*Cor.* 2. *Theor.* 8.) But fince $A^{\frac{1}{m}} = B$, therefore $\overline{A^{\frac{1}{m}}}^n = B^n$; confequently $\overline{A^n}^{\frac{1}{m}} = \overline{A^{\frac{1}{m}}}^n$.

SCHOLIUMS.

1. When $A^{\frac{1}{m}}$ is rational, fo alfo is $\overline{A^n}^{\frac{1}{m}}$, becaufe this is equal to $\overline{A^{\frac{1}{m}}}^{n}$, which is the Power of a rational Root: But tho' $\overline{A^n}^{\frac{1}{m}}$ is rational, it does not follow that $A^{\frac{1}{m}}$ is fo, as one Example fhews. Thus: Let $A = 3$, then $A^{\frac{12}{3}} = A^4 = 81$. But $A^{\frac{1}{3}}$ is Surd, for 3 has not a Cube Root.

2. Here we learn, that $A^{\frac{n}{m}}$ may indifferently be taken for the *n* Power of the *m* Root, or the *m* Root of the *n* Power of A, fince thefe are equal. So that henceforth we fhall take it either way, as fhall be moft ufeful. Hence alfo we have a Rule for expreffing the *Involution* of a fimple Root, or the *Evolution* of a fimple Power.

THEOREM XI.

If the Index of a mixt Power *(i. e.* which has a fractional Index) has both its Members, Numerator and Denominator, equally multiplied or divided, the Products or Quotes put in place of the others make an equivalent Expreffion, or exprefs a mixt (or fimple) Power or Root of the fame Number equal in Value to the former. Thus $\frac{4}{6} = \frac{2}{3}$, and therefore $A^{\frac{4}{6}} = A^{\frac{2}{3}}$. Univerfally, let $n = ab$, $m = ad$. Then is $A^{\frac{n}{m}}$ (or $A^{\frac{ab}{ad}}) = A^{\frac{b}{d}}$.

Example:

$A = 16777216$. $A^{\frac{1}{6}} = 16$

$A^{\frac{4}{6}} = 65536 = 16^4$

$A^{\frac{1}{3}} = 256$. $A^{\frac{2}{3}} = 65536$.

Demonftr. $A^{\frac{ab}{ad}}$ is the *ad* Root of the *ab* Power of A (by the Notation) which is the *d* Root of the *a* Root *(Theor. 9.)* of the *a* Power of the *b* Power *(Theor. 8)* Now the *b* Power of A is A^b, and the *a* Power of this is A^{ab}, *(Theo. 8.)* whofe *a* Root is A^b, *(Theor. 8. Cor. 2.)* and the *d* Root of this is $A^{\frac{b}{d}}$ *(per* Notation) *i. e.* $A^{\frac{ab}{ad}} = A^{\frac{b}{d}}$. Or the Demonftration will go on the fame way by taking $A^{\frac{ab}{ad}}$ for the *ab* Power of the *ad* Root.

THEOREM XII.

The Simple Power of a Mixt Power, or Mixt Power of a Simple, of any Number, is equal to a Mixt Power of the fame Number, whofe Numerator is the Product of the Simple Index by the Numerator of the Mixt one, and its Denominator that of the Mixt one; or, in fhort, whofe Index is the Product of the two given Indexes. Thus, the *r* Power of $A^{\frac{n}{m}}$ is $A^{\frac{rn}{m}}$; which is alfo the $\frac{n}{m}$ Power of the *r* Power; *i. e.* $\overline{A^r}^{\frac{n}{m}}$.

Example.

$A = 4096$. $A^{\frac{1}{4}} = 8$. $A^{\frac{3}{4}} = 512$

Squ. of $A^{\frac{3}{4}}$ or $A^{\frac{6}{4}} = 32768$

Demonftr. $A^{\frac{n}{m}}$ is the *n* Power of the *m* Root of A; therefore the *r* Power of $A^{\frac{n}{m}}$ is the *r* Power of the *n* Power of the *m* Root: But the *r* Power of the *n* Power is the *rn* Power, *(Theor.* VIII.) therefore the *r* Power of $A^{\frac{n}{m}}$ is the *rn* Power of the *m* Root, *i. e.* $A^{\frac{rn}{m}}$, *(per* Notation.) Again, This is alfo the $\frac{n}{m}$ Power of A^r: For the Index $\frac{n}{m}$ is the *m* Root of the *n* Power; wherefore $\overline{A^r}^{\frac{n}{m}}$ is the *m* Root of the *n* Power of the *r* Power; *i. e.* the *m* Root of the *nr* Power, or *nr* Power of the *m* Root; *viz.* $A^{\frac{nr}{m}}$.

THEO.

THEOREM XIII.

THE Simple Root of a Mixt Power, or Mixt Power of a Simple Root of any Number, is such a Mixt Root of the same Number, whose Denominator is the Product of the Simple Index into the Denominator of the Mixt one, and the Numerator that of the Mixt one. In short, whose Index is the Quote of the Mixt one, by the Name of the Simple one, or the Product of the two Indexes, taking the Simple Root fractionally: Thus the r Root of $A^{\frac{n}{m}}$, or the $\frac{n}{m}$ Root of $A^{\frac{1}{r}}$, is $A^{\frac{n}{mr}}$: For $\frac{n}{m} \div r = \frac{n}{m} \times \frac{1}{r} = \frac{n}{mr}$.

Example:

$A = 65536.$　$A^{\frac{1}{2}} = 256.$

$A^{\frac{3}{2}} = 16777216.$ Sq. Root of $A^{\frac{3}{2}}$, or $A^{\frac{3}{4}} = 4096.$

Demonstr. $A^{\frac{n}{m}}$ is the m Root of the n Power of A; therefore the r Root of $A^{\frac{n}{m}}$ is the r Root of the m Root of A^n, *i. e.* the rm Root, *(Theor.* 9.) or $A^{\frac{n}{rm}}$. Again, this is also the $\frac{n}{m}$ Root or Power of $A^{\frac{1}{r}}$. For $\frac{n}{m}$ expresses the n Power of the m Root, and so the $\frac{n}{m}$ Root of $A^{\frac{1}{r}}$ is the n Power of the m Root of the r Root, *i. e.* the n Power of the mr Root, or $A^{\frac{n}{rm}}$.

THEOREM XIV.

THE *mixt* Power of a *mixt* Power is equal to a *mixt* Power whose Index is the Product of the given Indexes. Thus, the $\frac{r}{s}$ Power of the $\frac{n}{m}$ Power is the $\frac{rn}{sm}$ Power, *i. e.* $\overline{A^{\frac{n}{m}}}^{\frac{r}{s}} = A^{\frac{nr}{ms}}$.

Demonstr. The r Power of $A^{\frac{n}{m}}$ is $A^{\frac{rn}{m}}$, *(Theor.* 12.) and the s Root of this is $A^{\frac{rn}{sm}}$ *(Theor.* 13.) which is therefore the s Root of the r Power, or the r Power of the s Root of $A^{\frac{n}{m}}$.

THEOREM XV.

THE Product of any two Roots Simple or Mixt, or of any Power and Root of the same Number, is equal to such a Power or Root of the same Number whose Index is the Sum of the given Indexes. Thus $A^{\frac{1}{n}} \times A^{\frac{1}{r}} = A^{\frac{1}{n} + \frac{1}{r}} = A^{\frac{r+n}{nr}}$; and $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m} + \frac{r}{s}} = A^{\frac{ns+mr}{ms}}$; Also, $A^n \times A^{\frac{r}{s}} = A^{n + \frac{r}{s}} = A^{\frac{ns+r}{s}}$.

Demonstr. The most universal Case or Expression is $A^{\frac{n}{m}} \times A^{\frac{r}{s}}$; for by supposing n or r, or each of them, equal to 1, you make them Simple Roots; and by making m or s equal to 1, you make that Term a Simple Power: And so all the Variety supposed in the Theorem will be demonstrated in this one Form; thus: The Thing to be demonstrated is, that $A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{n}{m} + \frac{r}{s}} = A^{\frac{ns+mr}{ms}}$: In order to which, suppose $A^{ns} = B$, and $A^{mr} = D$; then is $B^{\frac{1}{ms}} = A^{\frac{ns}{ms}}$ *(Ax.* 1.) $= A^{\frac{n}{m}}$ *(Theor.* XI.) Also $D^{\frac{1}{ms}} = A^{\frac{mr}{ms}} = A^{\frac{r}{s}}$; and $BD = A^{ns} \times A^{mr} = A^{ns+mr}$ *(Theor.* 6.) Hence $A^{\frac{ns+mr}{ms}} = BD^{\frac{1}{ms}} = B^{\frac{1}{ms}} \times D^{\frac{1}{ms}} = A^{\frac{n}{m}} \times A^{\frac{r}{s}}$.

T H E O R E M XVI.

T H E Quote of any two Roots, Simple or Mixt, or of any Power and Root of the fame Number, is equal to fuch a Power or Root of the fame Number, whofe Index is the Difference of the given Indexes, when the Index of the Dividend is greater than that of the Divifor. But if the Index of the Divifor is the greater, fet 1 over that Number fo found, or divide 1 by it, and this Fraction or Quote, is the Quote fought. Thus $A^{\frac{n}{m}} \div$

$$A^{\frac{r}{s}} = A^{\frac{n}{m} - \frac{r}{s}} = A^{\frac{ns - mr}{ms}} \text{ when } \frac{n}{m} \text{ is greater than } \frac{r}{s}; \text{ but it is } 1 \div A^{\frac{mr - ns}{m.s}} \text{ if } \frac{n}{m} \text{ is}$$

lefs than $\frac{r}{s}$. All other Cafes are contain'd in this Form.

Demonftr. Suppofe $A^{ns} = B$, and $A^{mr} = D$, then is $B^{\frac{1}{ms}} = A^{\frac{ns}{ms}} = A^{\frac{n}{m}}$, and $D^{\frac{1}{ms}} =$

$A^{\frac{mr}{ms}} = A^{\frac{r}{s}}$; alfo $B \div D = A^{ns} \div A^{mr} = A^{ns-mr}$, (*Theor.* VII.) Hence $A^{\frac{ns-mr}{ms}}$

$= \overline{B \div D}^{\frac{1}{ns}} = B^{\frac{1}{ns}} \div D^{\frac{1}{ns}} = A^{\frac{n}{m}} \div A^{\frac{r}{s}}$.

But if $\frac{n}{m}$ is lefs than $\frac{r}{s}$, then alfo is ns lefs than mr; and therefore $A^{ns} \div A^{mr}$ is a pro-per Fraction. Let us fuppofe $mr = ns + a$, or $mr - ns = a$, then is $A^{rs} \div A^{mr} = A^{ns} \div A^{ns+a}$; but $A^{ns} \times a = A^{ns} \times A^a$, (*Theor.* VI.) wherefore reduce the Fraction $A^{rs} \div A^{mr}$, or $A^{ns} \div A^{ns+a}$ to lower Terms by dividing both by A^{ns}, the new and equi-valent Fraction is $1 \div A^a = 1 \div A^{mr-ns}$; but $A^{ns} \div A^{mr} = B \div D$, which is therefore

$= 1 \div A^{mr-ns}$: Hence $1 \div A^{\frac{mr-as}{ms}} = \overline{B \div D}^{\frac{1}{ms}} = B^{\frac{1}{ms}} \div D^{\frac{1}{ms}} = A^{\frac{n}{m}} \div A^{\frac{r}{s}}$.

General S C H O L I U M *relating to the preceding* Theorems.

F R O M the preceding Theorems we have Rules for the Multiplication, Divifion, Invo-lution and Evolution of Numbers expreffed in Form of Powers or Roots of other Num-bers, *i. e.* for a more fimple and convenient Expreffion of the Products and Quotes, Powers and Roots; and the Subftance of the whole may be reprefented in four General Rules, which being particularly exemplified, will fhew in one fhort View all the preceding Theo-ry. And *obferve*, That in order to reduce it to fo few Rules, any Expreffion made of any Letter A, with any Index, Integral or Fractional, may be called a *Power* of A; for it's fuch a Power as the Numerator expreffes, of a certain Root expreffed by the Deno-minator.

R U L E I. Add the Indexes of any two Powers of the fame Number A, and the Sum is the Index of a Power equal to the Product of the other two.

R U L E II. The Difference of the Indexes of two Powers of A, is the Index of a Power equal to the Quote of the other two; minding, that if the Index of the Divifor is greateft, 1 is to be fet over the Power found, and that fractional Expreffion is the true Quote.

R U L E III. If the Index of a given Power of A is multiplied by another Index, the Product is the Index of a Power which is equal to fuch a Power of the given Power as

I   that

that other Index denominates. And becaufe the Word *Power* does here fignify both what is in a more particular Definition call'd *Power* and *Root*, therefore this Rule comprehends both *Involution* and *Evolution*.

RULE IV. If to the Product or Quote of two Numbers any Index is applied, the Expreffion is equal to the Product or Quote of the fame Powers of thefe two Numbers.

EXAMPLES *of thefe* Rules.

RULE 1.   *Theor.*

$$A^n \times A^r = A^{n+r} \quad \text{-----} \quad \text{6th.}$$

$$A^n \times A^{\frac{1}{r}} = A^{\frac{1+nr}{r}}$$

$$A^{\frac{1}{n}} \times A^{\frac{1}{r}} = A^{\frac{r+n}{nr}}$$

$$A^n \times A^{\frac{r}{s}} = A^{\frac{ns+r}{s}}$$

$$A^{\frac{n}{m}} \times A^{\frac{r}{s}} = A^{\frac{ns+mr}{ms}}$$

$$A^{\frac{1}{u}} \times A^{\frac{r}{s}} = A^{\frac{s+ur}{us}}$$

$$\left.\right\} \ 15\text{th.}$$

RULE 2.

$$A^n \div A^m = A^{n-m}, \text{ or } A^{\frac{1}{m-n}} \quad\quad 7\text{th.}$$

$$A^n \div A^{\frac{1}{r}} = A^{\frac{nr-1}{r}}$$

$$A^{\frac{1}{u}} \div A^{\frac{1}{r}} = A^{\frac{r-n}{nr}}, \text{ or } 1 \div A^{\frac{n-r}{nr}}$$

$$A^n \div A^{\frac{r}{s}} = A^{\frac{ns-r}{s}}, \text{ or } 1 \div A^{\frac{r-ns}{s}}$$

$$A^{\frac{n}{m}} \div A^{\frac{r}{s}} = A^{\frac{ns-mr}{ms}}, \text{ or } 1 \div A^{\frac{mr-ns}{ms}}$$

$$A^{\frac{1}{u}} \div A^{\frac{r}{s}} = A^{\frac{s-nr}{m}}, \text{ or } 1 \div A^{\frac{nr-s}{m}}$$

$$\left.\right\} \ 16\text{th.}$$

RULE 3.   *Theor.*

$$\overline{A^m}^n = A^{mn} \quad \text{------} \quad 8\text{th.}$$

$$\overline{A^{\frac{1}{m}}}^{\frac{1}{n}} = A^{\frac{1}{mn}} \quad \text{--------} \quad 9\text{th.}$$

$$\left. \begin{array}{c} \overline{A^{\frac{1}{m}}}^n \\ \overline{A^n}^{\frac{1}{m}} \end{array} \right\} = A^{\frac{n}{m}} \quad \text{------} \quad 10\text{th.}$$

$$\left. \begin{array}{c} \overline{A^{\frac{n}{m}}}^r \\ \overline{A^r}^{\frac{n}{m}} \end{array} \right\} = A^{\frac{nr}{m}} \quad \text{----} \quad 12\text{th.}$$

$$\left. \begin{array}{c} \overline{A^n}^{\frac{1}{r}} \\ \overline{A^{\frac{1}{r}}}^{n} \end{array} \right\} = A^{\frac{n}{rm}} \quad \text{-----} \quad 13\text{th.}$$

$$\overline{A^{\frac{n}{m}}}^r = A^{\frac{nr}{ms}} \quad \text{------} \quad 14\text{th.}$$

RULE 4.

$$A^n \times B^n = \overline{AB}^n \quad \text{----} \quad 1\text{ft.}$$

$$A^n \div B^n = \overline{A \div B}^n \quad \text{----} \quad 2\text{d.}$$

$$A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{AB}^{\frac{1}{n}} \quad \text{-----} \quad 3\text{d.}$$

$$A^{\frac{1}{n}} \div B^{\frac{1}{n}} = \overline{A \div B}^{\frac{1}{n}} \quad \text{----} \quad 4\text{th.}$$

$$A^{\frac{n}{m}} \times B^{\frac{n}{m}} = \overline{AB}^{\frac{n}{m}} \quad \text{----} \quad 5\text{th.}$$

$$A^{\frac{n}{m}} \div B^{\frac{n}{m}} = \overline{A \div B}^{\frac{n}{m}} \quad \text{----} \quad 5\text{th.}$$

THEOREM XVII.

If the *n* Power of one Number is equal to the *m* Power of another, then is the *m* Root of the firft equal to the *n* Root of the fecond: Thus if $A^n = B^m$, then is $A^{\frac{1}{m}} = B^{\frac{1}{n}}$.

Demonftr. Since $A^n = B^m$, then is $A^{\frac{n}{m}} = B$, *(Ax. 1.)* and $B^{\frac{1}{n}} = A^{\frac{n}{mn}}$, *(Ax. 1.)* $= A^{\frac{1}{m}}$, (*Theor* XI.)

The *Reverfe* is alfo true, *viz.* That if $A^{\frac{1}{m}} = B^{\frac{1}{n}}$, therefore $A^n = B^m$; for $A^{\frac{n}{m}} = B^{\frac{n}{n}} = B$, and therefore $A^n = B^m$.

SCHOLIUM.

SCHOLIUM. This Theorem may be made yet more univerſal by taking any Miſt Index, and making the Theorem thus: If $A^{\frac{n}{m}}=B^{\frac{r}{s}}$, then is $A^{ns}=B^{rm}$; and hence again, $A^{\frac{1}{rm}}=B^{\frac{1}{ns}}$: For the firſt, ſuppoſe $A^n=a$, and $B^r=b$, then is $A^{\frac{n}{m}}=a^{\frac{1}{m}}$ and $B^{\frac{r}{s}}=b^{\frac{1}{s}}$; wherefore $a^{\frac{1}{m}}=b^{\frac{1}{s}}$; and hence, (as is already ſhewn) $a^s=b^m$. But $a=A^n$, and $b=B^r$, therefore $a^s=A^{ns}$, and $b^m=B^{rm}$; that is, $A^{ns}=B^{rm}$; whence again, $A^{\frac{1}{rm}}=B^{\frac{1}{ns}}$.

COROLL. If $A^n=B^m$, and if $A^{\frac{1}{m}}$ is Rational, ſo alſo is $B^{\frac{1}{n}}$, ſince they are equal.

THEOREM XVIII.

IF $A^n=B^m$, call $m-n=d$. Then is A^d a Power of the Order m, and B^d a Power of the Order n; i. e. $A^{\frac{d}{m}}$, and $B^{\frac{d}{n}}$ are both Rational, tho' they are not always equal.

Demonſtr. Since $A^n=B^m=B^{n+d}$, (for $m=n+d$) multiply both by A^d; and then $A^{n+d}=B^{n+d}\times A^d$; therefore A^d is a Power of the Order $n+d$ or m, (by *Coroll.* 2 *Theor.* II) Divide both by B^d and it is $A^n \div B^d=B^n$; therefore B^d is a Power of the Order n, (*Coroll.* 3. *Theor.* II) i. e. $A^{\frac{d}{m}}$ and $B^{\frac{d}{n}}$ are both Rational.

COROLL. If $d=1$, that is, $m=n+1$, then A has an $n+1$ Root, and B an n Root. i. e. $A^{\frac{1}{m}}$ and $B^{\frac{1}{n}}$ are both rational; as they are alſo equal by the Theorem.

LEMMA.

If any Fraction $\frac{a}{r}$ is in its leaſt Terms, or is not ſo, then, accordingly, any Power of this Fraction, as $\frac{a^n}{r^n}$, is alſo in its loweſt Terms, or is not ſo. And the *Converſe*, if $\frac{a^n}{r^n}$ is or is not in its loweſt Terms, accordingly $\frac{a}{r}$ is or is not ſo.

The *Demonſtration* of this Truth muſt be referred to another Place, becauſe it depends upon Principles not yet explained: You'll find it demonſtrated in *Book* V. *Ch.* I. *Theor.* 1; *Coroll.* In the mean time we muſt ſuppoſe it to be true, for the ſake of ſome things belonging to this *Book*, whoſe Demonſtration depends upon this *Lemma*, and which could not be ſo regularly referred.

THEOREM XIX.

IF any Integer A has not a propoſed Root in Integers, it can have no determinate Root of that Order; i. e. it is a *Surd* Power of that Order. Or thus: If an Integer A is not the Power of a certain Order of an integral Root, it cannot be ſo of a fractional.

Example. 7 has no Square or Cube Root in Integers, and therefore has no ſuch determinate Root in a Fraction.

Demonſtr. Let $\frac{a}{r}$ be any Fraction in its loweſt Terms, and ſuch as is not equal to any Integer, i. e. let r be greater than 1; then, by the *Lemma*, $\frac{a^n}{r^n}$ is alſo a Fraction in its loweſt Terms: and conſequently, r^n is not an aliquot Part of A^n; nor, conſequently, is $\frac{a^n}{r^n}$ an Integer; for in this Caſe $\frac{a^n}{r^n}$ would not be in its loweſt Terms: Hence again it's clear, that no

no Fraction (fuch as is not equal to an Integer) can be any Root to a Whole Number: For fuppofe $\frac{s}{m}$ to be the n Root of A, then is $\frac{s^n}{m^n} = A$; but if $\frac{s}{m}$ is in its leaft Terms, fo is $\frac{s^n}{m^n}$; therefore $\frac{s^n}{m^n}$ not an Integer is equal to A an Integer, which is abfurd. Again, if $\frac{s}{m}$ is not in its leaft Terms, let $\frac{a}{r}$ be its leaft Terms; then, becaufe $\frac{s}{m} = \frac{a}{r}$ therefore $\frac{s^n}{m^n} = \frac{a^n}{r^n} = A$. But $\frac{a}{r}$ being in its leaft Terms, fo is $\frac{a^n}{r^n}$; whence the fame Abfurdity as before.

C H A P. II.

Containing the P R A C T I C E *of* Involution *and* Evolution.

§. I. Probl. *Of* I N V O L U T I O N, *or* Raifing *of* Powers.

THE *General Rule* for the Practice of *Involution* is plainly contained in the Definition, and is nothing elfe but a continued Multiplication of the Root into it felf, whereby, to come at any higher Power, we muft make up the Series of all the interior Powers: Thus; the Series of the Powers of 3 is 3 : 9 : 27 : 81 : 243, *&c.* By this Operation 3 × 3 × 3 × 3 × 3, *&c.*

But there is a particular Method whereby all Powers above the Cube or 3d Power may be found, without actually finding all the inferior Powers: Which Rule is this:

R U L E. Find, by the general Rule, two or more fuch Powers of the given Root as that the Sum of their Indexes be equal to the Index of the Power required; then multiply thefe Powers continually into one another; the Product is the Power fought. Or find any one Power whofe Index is an aliquot Part of the Index of the Power fought, and involve that Power to an Index equal to the Denominator of that aliquot Part.

Example 1. To find the 7th Power of 4, I find the 2d, 3d, and 4th Powers; *viz.* 16, 64, 256; then the 3d × 4th, or 64 × 256 = 16384 the 7th Power; becaufe 3 + 4 = 7. Or inftead of finding the 4th Power I find the 5th, by multiplying the 2d and 3d; *viz.* 16 × 64 = 1024; then the 2d × 5th, or 16 × 1024 = 16384 the 7th Power.

Example 2. To find the 12th Power of 3, I find the Square, *viz.* 3 × 3 = 9; the Square of the Square, or 9 × 9 = 81 is the 4th Power; again, the 2d × 4th, or 9 × 81 = 729 the 6th Power; then 6th × 6th, or 729 × 729 = 531441 the 12th Power.

I need infift no more on this Practice; the *Reafon* of which is plainly contained in the preceding *Theor.* VI. But I muft *obferve*, That it does not in every Cafe give any Advantage either of Eafe or Expeditioufnefs to the Work; yet as it will do fo in many Cafes, and in none can it make the Operation more tedious, it will always be a very convenient Method.

Of the Practice *in* Univerfal Characters.

As to the literal Practice, or Involution of Numbers reprefented by Letters and Indexes, this is alfo fufficiently explained in the preceding Definitions and Theore ms.

X

But

But *obferve*, That when a Root is reprefented as a complex Quantity; for Example, If inftead of S we put $A + B$, its Powers may be reprefented two different Ways; Thus,

$\overline{A + B}^2$ $\overline{A + B}^3$ or $\overline{A + B}^n$; which Method is in fome Cafes fufficient; but in others it's neceffary to have the Operation performed, and the Power expreffed according to the Refult of the Multiplication; fo that the Index be applied only to the fingle Letters: Thus,

$\overline{A + B}^2 = A^2 + 2AB + B^2$. The moft confiderable and important of all thefe Cafes of complex Roots, with their Powers, is that wherein there are only two Members in the Root, as $A + B$, called hence a *Binomial* Root; or $A - B$, called a *Refidual* Root; the Confideration of whofe Powers, *i.e.* of their Compofition by the various Powers and Multiples of the different Members of the Root, has been found of very great Ufe in Mathematicks, and in Arithmetick, efpecially for the Bufinefs of the Extraction of Roots; in order to which I fhall here explain it.

Of the Compofition *of the Powers of a* Binomial *and* Refidual *Root.*

In the annex'd Operation you fee the feveral Powers of $A + B$ raifed by Multiplying, according to the common Rules, each Member of the Root into each Member of the feveral Powers, which produces the next Powers; in which thefe Things are remarkable.

Obfervations *on the Table of* Powers.

I. In the Expreffion of each Power there are as many and no more different Members, (which contain different Powers of the Parts of the Root A and B) as the Index + 1 expreffes: For tho' each Member of any of the Powers being multiplied by A and B feparately, do make in all twice as many Products as the Terms in the Power multiplied; yet each of the Series of Products by A and B have all their Terms fimilar, exeept the firft Product by A, and the laft by B; for thefe two are A^n and B^n; *i.e.* the two Series of Products by A and B contain in their feveral Members the fame Powers of A and B, except the firft Term of the Line of Products by A, which is A^n, and the laft Term of the Line by B, which is B^n; confequently, thefe fimilar Products are reducible to a more fimple Expreffion by adding them together, (*i.e.* addding the Numbers by which they are multiplied, and joining the common or fimilar Letters with their Indexes) thus, $A^3 B + 3 A^3 B = 4 A^3 B$; alfo $3 A^2 B^2 + 3 A^2 B^2 = 6 A^2 B^2$; which explains the Reafon of placing the Lines of Products by A and B, as is here done, *viz.* in order to the Addition of fimilar Products. That this Obfervation will hold true however far the Powers are carried, is eafily feen from the Nature of the Thing; which will yet farther appear from the next Obfervation, in which we have a joint Demonftration of this.

II. Each Power of $A + B$ contains a Series of gradually different Powers of A and of B: Thus, The Index of any Power of $\overline{A + B}$ being *n*, the firft Term is fimply A^n, and the laft

TABLE *of Powers raifed from the Root* $A + B$.

Root. $A + B$

$A + B$

$\overline{A^2 + AB}$
$\underline{\qquad + AB + B^2}$

Square. $A^2 + 2AB + B^2$
$A + B$

$\overline{A^3 + 2A^2B + AB^2}$
$\underline{\qquad + A^2B + 2AB^2 + B^3}$

Cube. $A^3 + 3A^2B + 3AB^2 + B^3$
$A + B$

$\overline{A^4 + 3A^3B + 3A^2B^2 + AB^3}$
$\underline{\qquad + A^3B + 3A^2B^2 + 3AB^3 + B^4}$

4th Power. $A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$

&c.

laſt is B^n; the intermediare Terms containing each a different Power of both A and B, multiplied together; the Index of A decreaſing gradually by 1 in each Term from A^n to the Term preceding the laſt, or B^n, in which it is ſimply A; and the Indexes of B increaſing the ſame way from the Term next after A^n, in which it is only B, to the laſt, or B^n; ſo that in all the intermediate Terms there is ſome Power of A and of B; and the Sum of their Indexes is equal to n, the Index of the Power propoſed of $\overline{A+B}$. Therefore, omitting the other Numbers, which are Multipliers in the ſeveral Terms of any Power of a Binomial, whoſe Index is n, theſe Terms, in as far as they are compoſed of the Powers of A and B, may be repreſented thus;

$$A^n + A^{n-1} \times B + A^{n-2} \times B^2 + A^{n-3} \times B^3 + \&c. \; A^2 \times B^{n-2} + A \times B^{n-1} + B^n.$$

wherein there are as many Members as $n+1$, according to the firſt Obſervation; and the Index of B, or the Number taken from n in the Index of A, is the Number of Terms after A^n to any Term.

This *Obſervation* we ſee to be true ſo far as the Table of Powers is carried; and that it muſt be true for ever, is eaſy to perceive. Or it may be demonſtrated, *thus*: Suppoſe it's true in any one Caſe or Power of $\overline{A+B}$, as the n Power, it muſt be true in the next Caſe, or the $n+1$ Power: becauſe when each Term of the given Power is multiplied by

$$\begin{array}{l} A^n + \overset{n-1}{A} \times B + \overset{n-2}{A} \times B^2 + \overset{n-3}{A} \times B^3 +, \&c. + A \times \overset{n-1}{B} \times B^n \\ A+B \\ \hline \overset{n+1}{A} + A^n \times B + \overset{n-1}{A} \times B^2 + \overset{n-2}{A} \times B^3 +, \&c. + A^2 \times \overset{n-1}{B} + A \times B^n \\ + A^n \times B + \overset{n-1}{A} \times B^2 + \overset{n-2}{A} \times B^3 +, \&c. + A^2 \times \overset{n-1}{B} + A \times B^n + B^{n+1} \end{array}$$

A, the Products muſt have A once more involved in them than in the Term multiplied: And ſince the Indexes of A decreaſe gradually from A^n

in the Terms multiplied, conſequently they will decreaſe gradually from A^{n+1} in the Series of Products; the Powers of B continuing as they were. Again; The Series of Products made by B muſt have B once more involved in each; and conſequently increaſing gradually from B to B^{n+1}, leaving the Powers of A as they were: But again, Theſe Products made by B are all of them, (except the laſt B^{n+1}) ſimilar to the ſeveral Products, (after the firſt A^{n+1}) made by A; becauſe the Indexes of A in the given Power decreaſing from the firſt Term A^n, which has no Power of B multiplied into it, and thoſe of B increaſing from the ſecond Term $A^{n-1} \times B$ to the laſt Term B^n, it's plain that A multiplied into any Term except the firſt A^n, and B multiplied into the preceding, muſt make ſimilar Products; for A multiplied into any Term raiſes the Index of the Power of A by 1, which makes it equal to the Index of A in the preceding Term, without changing that of B; and B multiplied in the **preceding** Term, raiſes the Power of B in it to the Index of B in the following Term, without changing that of A; conſequently theſe Products, are ſimilar, which makes the thing obſerved manifeſtly true in any Caſe, in conſequence of its being true in the preceding: But it's true in the Root or 1ſt Power, and as far as we have carried the Powers, therefore it's univerſally true. And this alſo is manifeſt, that the Sum of the Indexes of A and B that are in any Term, is always equal to the Index of the Binomial Power, *viz. n.* Add alſo this Obſervation, that the Index of A or B is always 1 leſs than the Number of Terms from A^n or B^n, to that Term.

S c h o l i u m. If any one Member of a Binomial is 1, as $A+1$, then the Powers of 1 being all 1, the Powers of ſuch a Root will conſiſt only of the Series of the Powers of A, and 1 added; thus, $A^n + A^{n-1} + A^{n-2} + \&c. + 1$. Or thus, $1 + A + A^2 + \&c. + A^n$.

III. The Numbers which in every Power are multiplied into the ſeveral Terms are called *Coefficients* (*i e.* joint Multipliers or Factors) of theſe Terms; and from the Manner of raiſing

the Powers this is to be obferved, That the Coefficients of the firft and laft Terms are 1, and thofe of the intermediate Terms are each the Sum of the Coefficients of the correfponding and preceding Terms of the preceding Power; thus, The Coefficient of the third Term of the 4th Power is 6, equal to 3 + 3, the Coefficients of the 3d and 2d Terms of the 3d Power, (fee the preceding *Table of Powers.*) Now that you may perceive the Reafon of this, and that it muft continue fo for ever in all Powers, confider thefe two Articles:

(1.) The Products of the feveral Terms of any Power, made by A or by B, do not change the Coefficients of the Terms multiplied, becaufe A and B have no Coefficient but 1. Then

(2.) The fimilar Products made by A and by B are thefe, *viz.* The Product of the 2d Term, (of the Power multiplied) by A, and the Product of the 1ft Term by B; the 3d Term by A, and the 2d Term by B; and fo on. Which fimilar Products are added by the adding of their Coefficients, and annexing the fimilar Parts or Powers of A and B.

Now from thefe two Things the univerfal Truth of the Obfervation is manifeft; and the annex'd Table, fo far as it is carried on by this Rule, fhews the Series of Coefficients of any Power of a Binomial.

TABLE *of Coefficients of the Powers of a Binomial Root.*

		Coefficients.
1ft	1 : 1	
2d	1 : 2 : 1	
3d	1 : 3 : 3 : 1	
4th	1 : 4 : 6 : 4 : 1	
5th	1 : 5 : 10 : 10 : 5 : 1	
6th	1 . 6 : 15 : 20 : 15 : 6 : 1	
7th	1 · 7 : 21 : 35 : 35 : 21 : 7 : 1	
8th	1 : 8 : 28 : 56 : 70 : 56 : 28 : 8 : 1	
9th	1 : 9 : 36 : 84 : 126 : 126 : 84 : 36 : 9 : 1.	
	&c. &c.	

(Powers.)

COROLL. From the two laft Obfervations we learn a new and eafier Way than the common, for raifing any Power of a Binomial Root. Thus: take the Series of Products of the Powers of A and B, according to the fecond Obfervation; and to thefe apply the proper Coefficients, as they ftand in this Table; and if you have no fuch Table, you muft raife one, as far as the propofed Power; which being done by fimple Addition, is much eafier than the common Rule. Thus, for Example; The 4th Power of A + B is

$$A^4 + 4\,A^3\,B + 6\,A^2\,B^2 + 4\,A\,B^3 + B^4.$$

But, again, to make this yet eafier, fee the following Obfervation, and its Corollary.

IV. Any two Coefficients in the Series belonging to each Power are the fame Numbers, if they are taken at equal Diftances from the Extremes, (which have both 1) for the Coefficients increafe from the one Extreme to the Middle Term, where there is one Middle Term, and decreafe from that to the other Extreme by the fame Series by which they increafed; and if there are two Middle Terms, thefe are equal, and they decreafe upon each hand by the fame Numbers to the Extremes. The univerfal Truth of this is manifeft from the way of conftructing the Table: For being true in any Cafe, (as we fee it is as far as the Table is carried) it muft be true in the next Cafe or Power, and fo on for ever. And hence again, if we call the Place of any Term from the one Extreme *a*, the other Term whofe Coefficient is equal, is from the fame Extreme in the Place expreffed by $n - a + 2$ (*n* being the Index of the Power) For the whole Number of Terms is $n + 1$, by *Obferv.* 1. And that Term which is in the *a* Place, from the one Extreme, muft neceffarily be in the $n + 1 - a + 1 = n - a + 2$ Place from the other Extreme. And fince Coefficients at equal Diftances from the two Extremes are equal; hence it is, that reckoning them both from the fame Extreme, their Places are *a* and $n - a + 2$. Again, If we call the Index $n = a + b - 2$. (*i. e.* add 2 to the Index, and fuppofe the Sum $n + 2 = a + b$, whereby $n = a + b - 2$) then are the Coefficients equal which are in the *a* and *b* Places from

from the fame Extreme; for $b = n - a + 2$; and we have feen already that the Coefficients, in the a and $n - a + 2$ Places are equal.

In the laft Place, take Notice, That the 2d Term from either Extreme has for its Coefficient the Index of the Power.

C o r o l l. Hence, in making up the Table of Coefficients for any Power, as $\overline{A + B}^n$, when we are come to that Series which has as many Terms as $\frac{n + 1}{2}$ *i. e.* the half Number of Terms belonging to the propofed Power n, when that Number $n + 1$ is an even Number; or that has as many Terms as $\frac{n + 2}{2}$ when $n + 1$ is an odd Number; we need raife the following Series of the Table to no more Terms, till we come to the propofed Power n; and the remaining Terms of it are the fame with thefe already found, taken in a reverfe Order, as above explained. Thus: To find the Coefficients of the 9th Power, which has ten Terms; when you have arrived, in making the Table, at the 4th Power, which has five Terms, you need raife no more Terms in the following Series till you come to the 10th, and then make the remaining five Terms of it the fame with the preceding, in a reverfe Order. And for the 8th Power, which has nine Terms, you muft alfo have the Coefficients compleat to the 4th Power, which has five Terms; and when you come to the 8th, the remaining four are the fame with the firft four reverfely.

V. The preceding Obfervations were all obvious: But the following moft valuable Property of the Coefficients, in which we have a curious Rule for finding the Coefficients of any Power without regard to the preceding Powers, we owe to the happy Genius of the incomparable Sir Isaac Newton; which is this, *viz.*

Rule. The Coefficient of any Term is equal to the Product of the Coefficient of the preceding Term multiplied into the Index of A *in that preceding Term, and divided by the Number of Terms from* A^n *to that Term:* And becaufe the Coefficients of the firft and fecond Terms are always known, which are 1 and n, by *Obferv.* 4. therefore it is plain, that by this Rule the Series of Coefficients of any Power may be found independently of preceding Powers.

Exam. The Coefficient of the fourth Term of the eighth Power is 56, the Index of A in that Term is 5; then $56 \times 5 = 280$, and $280 \div 4 = 70$, which is the 5th Term.

In order to the *Demonftration* of this Rule, we fhall firft explain the *univerfal Expreffion* of it in Letters, which is this: Take the Index of the Power n, and make this Series of Factors, $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$, *&c.* carrying it to a Number of Terms equal to $n + 1$; and the firft Term or 1 is the Coefficient of the firft Term of the Power; $1 \times \frac{n}{1}$ or fimply n is the fecond Coefficient; $1 \times \frac{n}{1} \times \frac{n-1}{2}$ is the third Coefficient, and fo on, taking in always one Factor more at every Step, till you have all the Coefficients belonging to that Power, which are in Number $n + 1$. But, as is before obferv'd, having found them for the one half of the Terms, or to the middle Term, the other half is found alfo without any farther Operation.

Now that this is a true and juft Expreffion of the preceding Rule, will be plain from thefe Confiderations: 1. That the firft and fecond Terms are in all Cafes 1 and n. 2. That the Index of A decreafes continually from A^n the firft Term; A^{n-1} being the fecond, and fo on; whereby it is manifeft, that according to this Rule with that in *Obferv.* 2. the n Power of $A + B$ is reprefented as in the following Series; which is called

The Binomial Theorem.

$$\overline{A + B}^n = 1 \times A^n + 1 \times n \times A^{n-1} \times B + 1 \times n \times \frac{n-1}{2} \times A^{n-2} \times B^2 + 1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times A^{n-3} \times B^3$$

$$+ 1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times A^{n-4} \times B^4 +, \textit{\&c.}$$ which is carried to a Number of Terms

equal

equal to $n+1$; and then the laft Term will be B^n; the Index of A being o, whereby A is out of that Term; and the Coefficient is 1.

Before we come to the *Demonftration*, we muft obferve upon this Expreffion of the Rule for the Coefficients, that the Numbers taken from n in the Numerators are always 1 lefs than the Denominators; and thefe (which are alfo equal to the Index of B, or the Number taken from n in the Index of A) being in Arithmetical Progreffion increafing from 1, the Numerators are in Arithmetical Progreffion decreafing from n. Hence the Denominator of the laft Factor in each Coefficient is the Number of Factors after 1; and is alfo the Number of Terms after A^n to that Term; wherefore if the Denominator of the laft Factor of any Coefficient is called a, that Term is in the $a+1$ place from the beginning; or if it is in the a Place, that Denominator is $a-1$. Wherefore the Coefficient of the a Place of the n Power is $1 \times n \times \frac{n-1}{2}$, &c. carried on till the laft Factor, is $\frac{n-a+2}{a-1}$; or, make a the Place of the Term after the firft, *i. e.* the Number of Terms -1; then the Coefficient is $1 \times n \times \frac{n-1}{2}$, &c. to $\frac{n-a+1}{a}$. And if we take this Series of Factors backwards, it is $\frac{n-a+2}{a-1} \times \frac{n-a+3}{a-2}$, &c. to 1, when a is the Number of Terms; or $\frac{n-a+1}{a} \times \frac{n-a+2}{a-1}$, &c. to 1, when $a+1$ is the Number of Terms. We fhall next demonftrate the univerfal Truth of this Rule for Coefficients. Thus:

Demonftration of the preceding Rule for COEFFICIENTS.

1. If the Rule is good in any one Cafe or Power of $A+B$, as the n Power, it muft therefore be good in the next Power, or $n+1$. To prove this Connection, fee the two following Series; whereof the firft contains the Coefficients for the n Power, according to the Rule; and the other the Coefficients for the $n+1$ Power, according to the fame Rule; and becaufe 1 does not multiply, I have omitted it in all the Coefficients but the firft, which is itfelf 1.

For the n Power, $1, \ \frac{n}{1}, \ \frac{n}{1} \times \frac{n-1}{2}, \ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c.

For the $n+1$ Power, $1, \frac{n+1}{1}, \frac{n+1}{1} \times \frac{n}{2}, \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3}, \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3} \times \frac{n-2}{4}$, &c.

By *Obferv.* 3. the Coefficients of any Power of $A+B$ are each equal to the Sum of the Coefficients of the correfponding and preceding Terms of the preceding Power of $A+B$; wherefore the firft Series being the true Coefficients of the n Powers, the fecond will be the true Coefficients of the $n+1$ Power; providing that they have this Connection with the former, *viz.* that any Term is the Sum of the correfponding and preceding Terms of that former; which is therefore the thing to be proved. Thus:

The firft Coefficient in all Powers is 1; then the Sum of the firft and fecond Coefficients of the n Power is $1 + \frac{n}{1} = \frac{n+1}{1}$ the fecond Term of the fecond Series. Again, the Sum of the fecond and third Terms of the firft Series is $\frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} = \frac{n}{1} \times \overline{1 + \frac{n-1}{2}}$ $= \frac{n}{1} \times \frac{n+1}{2} = \frac{n+1}{1} \times \frac{n}{2}$ (by changing the Order of the Numerators, which does not change the Product) and this is the third Term of the fecond Series; then the Sum of the

the third and fourth Terms of the firft Series is $\frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{n}{1} \times \frac{n-1}{2}$

$\times 1 + \frac{n-2}{3} = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n+1}{3} = \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3}$ (by changing only the Order of the Numerators, which does not change the Product) and this is the fourth Term of the fecond Series. From the Nature of thefe Series, it's evident they muft have, every-where, the fame Connection; or we may alfo fhew the Univerfality of it. Thus:

By what's fhewn in the Obfervation made upon the Expreffion of this Rule, the Coefficient of any Term, as that in the *a* Place after the firft or A^n, may be thus expreffed,

$1 \times \frac{n}{1} \times \frac{n-1}{2} \times,$ &c. $\times \frac{n-a+2}{a-1} \times \frac{n-a+1}{a}$, and the preceding Term will be $1 \times \frac{n}{1} \times \frac{n-1}{2} \times$

&c. $\times \frac{n-a+2}{a-1}$, which contains all the Factors of the other except the laft. The Sum

of thefe two is therefore, $1 \times \frac{n}{1} \times \frac{n-1}{2} \times,$ &c. $\times \frac{n-a+2}{a-1} \times \overline{1 + \frac{n-a+1}{a}} = 1 \times \frac{n}{1} \times$

$\frac{n-1}{2} \times,$ &c. $\times \frac{n-a+2}{a-1} \times \frac{n+1}{a} = 1 \times \frac{n+1}{1} \times \frac{n}{2} \times \frac{n-1}{3} \times,$ &c. $\times \frac{n-a+2}{a}$, (by chan-

ging the Order of the Numerators) which is the *a* Term after the firft in the $n+1$ Power; becaufe the Denominators are the fame Series, 1, 2, 3, &c. to *a*, which are the Denominators in all Powers; and the Numerators decreafe gradually from the Index $n+1$; fo that the Number fubtracted from the Index $n+1$ in the laft, is lefs by one than the Denominator, (as has been obferved and explained upon this Expreffion of the Rule); for the laft Numerator is here $n-a+2 = \overline{n+1} - a + 1 = \overline{n+1} - \overline{a-1}$.

2. But this Rule is true when applied to the firft Power, and to all the Powers as far as we have raifed them in the preceding Table; therefore, by what's now fhewn, it's true in the next Power above; and confequently in all above, *i, e.* in all the Powers whatever of $A + B$.

SCHOLIUMS.

1ft, Different Authors have made different Demonftrations of this Rule; I have chofen what I think as eafy as any of them, and fitteft for this place. In *Book* V. *Chap.* 6. you'll find another Demonftration of it from Principles which have not, as I know, been applied to this purpofe.

2. If inftead of a Binomial $A + B$ we take a Refidual $A - B$, it's manifeft that all the Difference betwixt its Powers and thefe of $A + B$ will be, That whereas all the Members of the Binomial Powers are added together, thefe of the Refidual Powers will be connected with the Signs of Addition and Subftraction, alternately; but the Powers of A and B, with

the Coefficients are the very fame: Thus, $\overline{A-B}^2 = A^2 - 2AB + B^2$, and $\overline{A-B}^3 = A^3 - 3A^2B + 3AB^2 - B^3$; alfo $\overline{A-B}^4 = A^4 - 4A^3B + 6A^2B^2 - 4AB^3 + B^4$, and fo on.

3. In applying this Rule for finding any Coefficient, (either of a Binomial or Refidual) obferve to take its Place from the neareft Extreme, A^n or B^n, which will make the Operation fhorter, and produce the fame Number, fince the Coefficients are the fame Series of Numbers, from either Extreme. Thus, to find the Coefficients in the *a* Place (from either Extreme) of the *n* Power: Compare *a*, and $n - a + 2$ (for, by *Obferv.* 4. the Coefficients in the *a*, and $n - a + 2$ Places, from the fame or different Extreme, are equal.) Which ever of thefe Numbers is leaft, find the Coefficient for that Place. *Example:* To find the 7th Coefficient of the 10th Power; I find the 5th Coefficient, which is equal to the 7th; for if $n = 10$, $7 = a$, then is $n - a + 2 = 5$.

4. Tho'

4. Tho' we had taken no notice of the Equality of Coefficients at equal Diftances from the two Extremes, yet the Rule now demonftrated would have fhewn it of itfelf: Thus, The Theorem for Coefficients is $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times, \&c. \frac{n-a+1}{a}$; which is the Coefficient in the $a+1$ Place, or the $\bar{a}$ Place after the firft. Now the Numerators decreafe from n to $n-a+1$, or $n-\bar{a}-1$, by a conftant Difference 1, and the Denominators increafe from 1 to $\bar{a}$. Again; Since the Number of Terms in the n Power is $n+1$, and in every Coefficient there are as many Factors as the Number of Terms from the Beginning, or A^n; therefore, if we want the laft Coefficient, or that in the $n+1$ Place, then is $a=n$; and confequently, $a+1=n+1$, and $n-a+1=n-n+1=1$; fo that the Numerators and Denominators are the very fame Series of Numbers, only in different Order, which alters not the Product; and being equal, therefore the Product is $=1$. Let us now exprefs the Series thus; $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \&c.$

$\frac{3}{n-2} \times \frac{2}{n-1} \times \frac{1}{n}$; it's plain the laft Coefficient but one, is the Product of this Series, excluding $\frac{1}{n}$; which Product is equal to $1 \times \frac{n}{1}$, for all the other Factors upon each hand of the middle one, (whofe Numerator and Denominator muft be equal) are reciprocal to one another, and fo make the Product of them all only 1; or if there are two middle ones they are Reciprocals. By the fame Reafon, the Coefficient in the laft Place but two, is is $1 \times \frac{n}{1} \times \frac{n-1}{2}$: For excluding $\frac{2}{n-1} \times \frac{1}{n}$, the middle Terms after $\frac{n-1}{2}$ deftroy one another's Effect, and make their total Product no more than 1: The fame Reafoning holds in every Place. And hence again obferve, that if we apply the Rule to find a Coefficient ftanding from the firft Place further than the middle Place, or the laft of two middle Places; then whenever in writing down the Factors, we come to one whofe Numerator and Denominator are equal, or to two adjacent Factors that are Reciprocals, there we may ftop; for what follows will deftroy the Effect of as many of thefe preceding that one whofe Numerator and Denominator are equal, or the firft of thefe two adjacent Reciprocals, as the remaining Number to be yet fet down; and therefore, by cutting off fo many of the Factors (as leaves a Number equal to the Place of the Coefficient fought, number'd from the neareft Extreme) we have what's fought: Thus; For the 8th Coefficient of the 10th Power, it is $1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3} \times \frac{7}{4} \times \frac{6}{5} \times \frac{5}{6} \times \frac{4}{7}$; which is $= 1 \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3}$, for the reft deftroy one another, being Reciprocals.

5. If we take the perpendicular Columns of the Table of Coefficients, it's plain thefe are Coefficients all in the fame Place, or Diftance from the Beginning in different Powers; and may be called *Similar* Coefficients of different Powers. Again; We have explained above, that if the Place of any Coefficient is a, the laft Factor that compofes it is $\frac{n-a+2}{a-1}$, and fo that Coefficient will be $1 \times \frac{n}{1} \times \frac{n-1}{2} \times, \&c. \times \frac{n-a+2}{a-1}$; then by changing the Value of n this will exprefs all the fimilar Coefficients in the a Place of different Powers: obferving this, That the loweft Value we can put upon n is $a-1$; becaufe no Power below that of the Order $a-1$ can have a Number of Terms equal to a (by *Obferv.* 1.); and if $n=a-1$, the Coefficient will be 1; for it is the a Coefficient of the $a-1$ Power, which being the laft Coefficient, is therefore 1, and is confequently the firft Term of the Series of fimilar Coefficients of different Powers, from that whofe Index is $a-1$: So that by taking n fucceffively equal to $a-1$, a, $a+1$, $\&c.$ we fhall have the Series of Coefficients of the $\bar{a}$ Place of thofe different Powers whofe Indexes are $a-1$, a, $a+1$, $a+2$, $\&c.$

But

But we may exprefs this Rule alfo thus: Inftead of n put $a + b - 2$, and it is

$$1 \times \frac{a + b - 2}{1} \times \frac{a + b - 3}{2} \times \&c. \text{ to } \frac{\overline{a + b - 2} - \overline{a - 2}}{a - 1} \text{ or } \frac{b}{a - 1};$$

which, according to the general Rule of Coefficients, is the Coefficient of the $\bar{a}$ Term of the $a + b - 2$ Power; and by taking b fucceffively equal to 1, 2, 3, $\&c.$ we fhall have hereby the Series of Coefficients in the $\bar{a}$ Place of all Powers from the $a - 1$ Power; for if $b = 1$, then is $a + b - 2 = a - 1$, and the Rule gives the firft fimilar Coefficient, which is always 1; if $b = 2$, then $a + b - 2 = a$, and we have the fecond fimilar Coefficient; if $b = 3$, then $a + b - 2 = a + 1$, and we have the third fimilar Coefficient, and fo on:

Or if we take $m = a + b$, the Rule is $1 \times \dfrac{m - 2}{1} \times \dfrac{m - 3}{2} \&c. \times \dfrac{m - a}{a - 1}$.

Hence again we have this general Truth to obferve, *viz* That the a Coefficient of any Power whofe Index is $a + b - 2$, is the fame as the b Term of the Series of fimilar Coefficients which are in the $\bar{a}$ Place of different Powers.   And this will eafily be proved from thefe two Confiderations : The 1ft is what we have already explained, *viz.* That if the Index of any Power is $a + b - 2$, then the a Coefficient of that Power is equal to its b Coefficient (See *Obferv.* 4.)   The 2d is, That from any Term in the Table of Coefficients, (*i. e.* any Coefficient of any Power) there ftand as many Terms on the right hand, as there are Terms above it in the perpendicular Column of fimilar Coefficients; and therefore that Term is in the fame Place of the fimilar Coefficients, and of the Line of Coefficients of that Power, numbering from the right hand : Wherefore it's plain, that the a Coefficient (reckoning from the left hand) of the $a + b - 2$ Power, is the fame as the b Term of the Column of fimilar Coefficients in the a Place of different Powers, becaufe it's the fame as the b Coefficient from the Right of the fame Power.

C O R O L L A R I E S.

1. Thefe Expreffions of Powers of a *Binomial* Root fhew us how the Difference betwixt any two fimilar Powers is compofed of the various Powers and Multiples of any one of the Roots, and the Difference betwixt the Roots: Thus, A being one Root, and B the Difference of that and another $A + B$, or $A - B$, the Difference of their Squares is $2AB + B^2$.   Hence

Having any Power of any Root, we can find another fimilar Power whofe Root fhall differ from the given one by any Difference, and that without either knowing or enquiring what that other Root is.

For *Example :* 144 is the Square of 12; and if the Difference betwixt this Root and another is 9, hence the Square of that other is $144 + 2 \times 12 \times 9 + 81 = 144 + 216 + 81 = 441$, if 12 is the leffer Root; but it is $144 - 216 + 81 = 9$, if 12 is the greater Root.

Obferve, If the given Difference B is 1, then in all the Terms wherein there is any Power of B, we have nothing but the Powers of A, with the Coefficients; except the laft Term B^n, which ftands alone, and is 1; for $\overline{A + 1}^2 = A^2 + 2A + 1$, and $\overline{A + 1}^3 = A^3 + 3A^2 + 3A + 1$.

2. We have here alfo learnt another Way of Raifing a given Number to any Power, by means of the fimilar Powers of the Binomial: Thus;

Take all the fignificant Figures of the given Number in their compleat Value, as fo many different Members that compofe it, by Addition; then take the two higheft, calling them A and B: Raife this Binomial to the propofed Power; then confider the firft two Members as one; call their Sum again A, and call the next Member B, and raife this new Binomial to the fame Power; in doing of which, *obferve*, that fo much of the Work is already done, becaufe the n Power of the firft Member of the prefent Binomial is the total Power of the preceding Binomial, which is already found; fo that what remains is to

make

make up the other Members of the Power fought, according to the general Canon. In the fame manner confider the three higheft Members as one, and call it A, and join the next Member, calling it B, and raife this Binomial; and thus proceed till all the Members are taken in.

Example: To find the Square of $246 = 200 + 40 + 6$, the Operation is

$$\overline{200 + 40}^2 = 40000 + 16000 + 1600 = 57600$$
$$\overline{A + B}^2 = A^2 + 2AB + B^2$$

Then $\overline{240 + 6}^2 = 57600 + 2880 + 36 = 60516$ the Square fought.
$$\overline{A + B}^2 = A^2 + 2AB + B^2$$

If there are more Members, you muft go on the fame Way.

Example 2. To find the Cube of $235 = 200 + 30 + 5$. the Work is

$$\overline{200 + 30}^3 = 8000000 + 3600000 + 540000 + 27000 = 12167000$$
$$\overline{A + B}^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

Then $\overline{230 + 5}^3 = 12167000 + 793500 + 17250 + 125 = 12977875$ the Cube fought.
$$\overline{A + B}^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

SCHOLIUM. As to this Method of raifing Powers, it's more tedious than the common Way, and therefore not to be recommended for Practice; the Defign of confidering it here being only for the fake of a particular Illuftration to be made by it of the Rules of Extraction.

§. II. *Of* EVOLUTION, *or* Extraction *of Roots.*

PART I. *Of* Whole Numbers.

Problem I. *To extract the* Square Root *of a* Whole Number.

RULE I. IN Order to the Solution of this Problem, we muft have a Table of fimple Squares, or Squares of Numbers from 1 to 9, as here in the Margin: Then

II. Beginning at the Right hand, diftinguifh the Figures of the given Number into Periods of two Figures as long as you can, by putting a Point over the firft Figure, and over every other Figure, *i. e.* paffing one, take the next.

Example: 1849 is pointed thus, 1849, and 34968 thus, 34968; the pointed Figure being the firft of each Period, and that on its Left the other; tho' the laft Period may fometimes have but one Figure.

The given Number being thus pointed, the Number of Points or Periods fhews us how many Figures the Root confifts of; to find which we proceed thus:

III. Take the laft Period, (or that next the Left) and feek it, or the next leffer Number you can find, in the Table of fimple Squares, the Root of this is the firft Figure on the Left of the Root fought; which being written down to the Right of the given Number, as we do the Quote in Divifion, then fet down its Square under the laft Period, and take their Difference, to which prefix the next Period of the given Number: And all this taken for one Number

Roots.	Squares.
1,	1
2,	4
3,	9
4,	16
5,	25
6,	36
7,	49
8,	64
9,	81

ber, as it ftands, we call the Second Refolvend. becaufe out of it we feek the next Figure of the Root, (the laft Period being the firft Refolvend) thus:

IV. Confider the Figure found as having o before it, and then multiply it by 2, (which is in effect, multiplying it by 20); make this Product a Divifor, and find how oft it is contained in the Refolvend; which, to the prefent purpofe, muft not be taken above nine times; tho' in fome Cafes it may be oftner contained; and then alfo it muft be under this Limitation, *viz.* that the Square of the Quote, or Figure now fet in the Root, added to its Product by the Divifor, the Sum do not exceed the Refolvend: Or, which is the fame thing, put the Quote, or fuppofe it put, in the Place of the o which ftands in the Place of Units of the Divifor; then multiplying the whole by the Quote, the Product muft not exceed the Refolvend: For if it do, the Figure taken is too great, and you muft try a leffer, till it anfwer. The Figure thus found is the next Figure of the Root fought, which muft be fet on the Right of the laft: And fetting the Sum or Product mention'd under the Refolvend, take their Difference, to which prefix the next Period of the given Number; and all this confidered as one Number, is your next Refolvend; out of which the next Figure is to be fought thus:

V. Take both the Figures of the Root found, as they ftand, for one Number; double it, and prefix o, (or prefix o, and then double, which is the fame thing,) and this is your Divifor: Find how oft it is contained in the Refolvend, under the fame Limitations as formerly; place the Figure found on the Right of thefe before found, and fubftracting the Product directed to be compared with the Refolvend from it; to the Difference prefix the next Period, and you have the next Refolvend; to which make a Divifor out of the Figures of the Root already found, the fame way as before; and thus proceed till all the Periods are employed, finding a new Figure of the Root for every Period: And if at any Step the Divifor is greater than the Refolvend, or if 1 added to the Divifor makes the Sum greater than the Refolvend; then place o in the Root, and prefix another Period, forming a new Divifor by fetting another o to the former Divifor, and fo go on.

E x a m p l e s.

Ex. 1. To find the Square Root of 1369, it is 37, as found by this

Operation.

$$
\begin{array}{c}
\overset{.}{1}\overset{.}{369} \left(\begin{array}{c} a. \ b \\ 3 \ 7 \end{array}\right. \\
a^2 = 9 \\
\hline
2a=60 \left.\begin{array}{c} \\ \end{array}\right) 469. \ \text{2d Refolvend.} \\
(a=30) \left.\begin{array}{c} \\ \end{array}\right) 469 = 2ab+b^2 \\
\hline
000 \\
\hline
\end{array}
$$

Explication.

The given Number being pointed, the laft Period is 13, and the next Square to this is 9, whofe Root is 3, which is the laft Figure of the Root; and calling it a, I take $a^2 = 9$ out of 13, and to the remaining 4 I prefix the next Period 69, which makes 469 the 2d Refolvend: Then taking $a = 30$, I double it, and make $2a = 60$ a Divifor; and feeking how oft it is contained in 469, under the Limitations of the Rule, I find it 7 times, which is therefore the other Figure of the Root; which is proved by this, that $2ab+b^2 = 469$, the Refolvend; and becaufe there is no Remainder, the given Number 1369 is a true Square, whofe Root is 37.

Example

Example 2. To find the Square Root of 23097636, it is 4806.

Operation.

$$a^2 = \begin{array}{r} \dot{2}3\dot{0}9\dot{7}\dot{6}\dot{3}\dot{6} \\ 16 \end{array} \left(\begin{array}{l} abbb \\ 4806 \\ \overline{c} \\ \overline{a} \end{array} \right.$$

$$\begin{array}{l} 2a = 80 \\ (a = 40) \end{array} \Big) \quad \begin{array}{l} 709. \text{ 2d Refolvend.} \\ 704 = 2ab + b^2 \ (b = 8) \end{array}$$

$$\begin{array}{l} 2a = 960 \\ (a = 480) \end{array} \Big) \quad 576. \text{ 3d Refolvend.}$$

$$\begin{array}{l} 2a = 9600 \\ (a = 4800) \end{array} \Big) \quad 57636. \text{ 4th Refolvend.}$$

$$57636 = 2ab + b^2 \ (b = 6)$$

$$00000$$

Explication.

The laſt Period being 23, the next Square leſſer is 16, whoſe Root is 4, which I place in the Root, and call-ing it *a*, I take $a^2 = 16$ out of 23, and to the Remainder 7 I prefix the next Period 90, which makes 709 the 2d Re-ſolvend: Then taking $a = 40$, according to the Rule, I double it, and $2 \times 40 = 80$ is the Diviſor, which is contain'd in the Reſolvend 8 times; which Number alſo anſwers the Limitations of the Rule: For 88×8 is $= 704 = 2ab + b^2$, or $\overline{2a + b} \times b$, *b* being 8, wherefore 8 is the next Figure of the Root ſought; and ſub-ſtracting 704 from 709, to the Remainder 5 I prefix the next Period 76, and 576 is the 3d Reſolvend; then taking $a = 480$, its Double, 960, is the Diviſor; which being greater than the Reſolvend, I ſet o in the Root, then prefixing the next Period 36, the 4th Re-ſolvend is 57636, and the Diviſor is 9600, (*a* being here 4800) which is the former Di-viſor with o prefix'd: Then I find 9600 contained in 57636, 6 times, which Number an-ſwering the Limitation of the Rule, I ſet 6 in the Root, and calling it *b*, I find $2ab + b^2 = 57636$, the Reſolvend; ſo that nothing remains; And ſo the given Number 23067636 is a perfect Square, whoſe Root is 4806.

S C H O L I U M S.

1. If you begin your Gueſs or Trial for the Quote in any of the Steps after the firſt, at the greateſt Number of times, not exceeding 9, that the Diviſor is contain'd in the Re-ſolvend: Then the Limitation of the Rule for the Number to be compared with the Re-ſolvend is ſufficient to determine when we have the true Figure; becauſe if that Number is greater than the Reſolvend the Quote is taken too big; and then we try the next leſſer, till it anſwer the Rule: Yet *obſerve*, that if you ſhould make trial at random, then tho' $2ab + b^2$ be leſs than the Reſolvend, yet the Quote *b* may be too little, and you'll prove it by this Mark, *viz.* The Remainder, after taking $2ab + b^2$ out of the Reſolvend, may be greater than the Diviſor, but it muſt not exceed the Sum of the Diviſor and double the Quote, *i. e.* $2a + 2b$, elſe the Quote is too little: *See below the* Explication *of* Exam. 3. And here I muſt *obſerve*, That ſome Authors think the forming of a Diviſor an uſeleſs thing, and would have us left altogether to a random Gueſs for the Figure of the Quote at every Step after the firſt, tho' they preſcribe the ſame neceſſary Limitation of the Figure gueſſed: But they have not conſidered this Conveniency of the Diviſor, that the greateſt Number of times it is contained in the Reſolvend not exceeding 9, is a Limit to our gueſſing; for the Figure ſought cannot exceed that, and ſo will in many Caſes ſave the Trouble of gueſſing at Figures which cannot anſwer. Beſides, the Diviſor is of a neceſſary Conſideration in the Demonſtration of the Rule; and a further Uſe of it, ſee in the next Article.

2. If the Diviſor is contained in the Reſolvend oftner than 9 times in any Step after the 2d, the Figure ſought is certainly 9: And alſo in the 2d Step it's 9, if the firſt Figure is at the ſame time, 5, 6, 7, 8, or 9. But if this is below 5, we muſt make trial; for ſome-times it will be 9, and ſometimes not. *See* Exam. 3, 4.   3. For

3. For forming the Divifor a little more eafily, you have no more to do after the 2d Step but add the double of the Figure laft found to the laft Divifor, and then prefix o; as you may eafily perceive in the Examples.

4. If there is a Remainder after all the Periods are employed, then the given Number is not a perfect Square; and the Root found is the Root of the greateft Integral Square contained in it. How to find a Mixt Root whofe Square fhall be within any affigned Difference from the given Number, fhall be taught in its proper Place.

Example 3. To find the Square Root of 151426. By the *Operation* we find it's not a Square, but the Root of the greateft Integral Square contained in it is 380.

Operation.

$$151426 \quad (389$$
$$a^2 = \quad 9$$
$$a$$
$$\overline{\hspace{2cm}}$$
$$2 \times 30 = 60) \quad 614$$
$$544 = 2ab + b^2$$
$$a$$
$$\overline{\hspace{2cm}}$$
$$2 \times 380 = 760) \quad 7026$$
$$6921 = 2ab + b^2$$
$$\overline{\hspace{2cm}}$$
$$105. \text{ Remainder.}$$

Explication.

Here in the fecond Step 60 is contained in 614, 10 times; yet the true Quote, or Figure for the Root, is only 8: For 9 would make the Product 621, which is greater than 614: And had we taken 7, it would have been found too little, from the Mark given in *Schol.* 1.; for then the Product is 499, which taken from 614 leaves 145, which is greater than 60 + 16 (or $2a + 2b$) = 76. And becaufe the Remainder of the whole Work is 105, the greateft Square contained in 151426 is 151321, whofe Root is 389.

Example 4. To find the Square Root of 15437052: The Root of the greateft Square contained in it is 3929.

Operation.

$$15437052 \quad (3929$$
$$a^2 = 9$$
$$a$$
$$\overline{\hspace{2cm}}$$
$$2 \times 30 = 60) \quad 643. \text{ 2d Refolvend.}$$
$$621 = 2ab + b^2.$$
$$a$$
$$\overline{\hspace{2cm}}$$
$$2 \times 390 = 780) \quad 2270. \text{ 3d Refolvend.}$$
$$1564 = 2ab + b^2.$$
$$a$$
$$\overline{\hspace{2cm}}$$
$$2 \times 3920 = 7840) \quad 70652 \text{ 4th Refolvend.}$$
$$70641 = 2ab + b^2.$$
$$\overline{\hspace{2cm}}$$
$$11 \text{ Remainder.}$$

Explication.

In the fecond Step, 60 is contained in 643 10 times, and the true Quote is 9. In the third Step, 780 is contained in 2270 only 2 times; and 7840 in 70652 9 times: The Remainder of the Operation being 11. So that the greateft Square contained in the given Number is 15437041, whofe Root is 3929.

Demonftration of the preceding Rule.

In order to this Demonftration, the following *Lemma's* muft be firft demonftrated.

L E M M A I.

The Product of any two Numbers can have at moft but as many Places of Figures as are in both the Factors; and at leaft but one Place fewer. *Exam.* 3 × 4 = 12, and 2 × 16. = 32.

Demonftr.

Demonſtr. 1. That the Product may have as many Places as both the Factors, one *Example* is enough to demonſtrate. Thus, $46 \times 82 = 3772$; and that in no Caſe it can have more, I thus prove·

Let any two Numbers be A, B; then take D the leaſt Number poſſible, which has one Place more than B; it's evident from the Notation of Numbers, that D will conſiſt of 1, with as many o's as the Number of Places in B; and alſo D will be a greater Number than B; if we then multiply A by D, the Product A D will be equal

$$A = 4678. \quad B = 37549$$
$$A \times B = 175654222$$
$$D = 100000. \quad A \times D = 467800000.$$

to A, with as many o's before it as are in D, *i. e.* as the Number of Figures in B: therefore it has as many Places, and can have no more than are in both A and B. But again; ſince B is a leſſer Number than D, therefore A B is a leſſer Number than A D, and conſequently cannot have more Places, *i. e.* more than are in A and B both.

2. The Product may have fewer Places than are in A and B both, which one Example will ſhew. Thus, $23 \times 346 = 7958$: but it can in no Caſe have above one Place fewer, which is thus proved.

$$A = 23. \quad B = 346$$
$$A B = 7958$$
$$D = 100. \quad A D = 2300.$$

Take any two Numbers A, B; and take D conſiſting of 1, with as many o's before it as the Figures leſs than one in B; *i. e.* the leaſt Number poſſible, which has as many Figures as B; then will the Product A D be equal to A, with as many o's before it as are in D, which are one fewer than the Figures in B; conſequently A D has as many Places, and can have no fewer than the Sum of the Places in A, and one fewer than are in B; *i. e.* all the Places in A D can be but one fewer than the Sum of thoſe in A and B. But ſince B is a greater Number than D, ſo will A B be greater than A D; and conſequently cannot have fewer Places than A D, which can be but one fewer than in A and B both.

C o r o l. A Number being multiplied into itſelf, the Product or Square cannot have more Places than double the Places of the Root; and but one fewer at leaſt than that double. Wherefore a Square being diſtributed into Periods, as the Rule directs, the Root has preciſely as many Figures as the Square has Periods.

L E M M A II.

If any Number A is not a Square, yet being diſtributed into Periods, according to the Rule, the greateſt Square contained in it, as N^2, will have preciſely as many Periods as that Number A has.

Exam. 237694 is not a Square, and the greateſt Square contained in it is 237169; both which have three Periods.

Demonſtr. 1. N^2 cannot have more Periods than A; for then it will have more Figures, and conſequently be a greater Number than A; contrary to Suppoſition.

$$A = 37694$$
$$N^2 = 237169$$
$$B = 10000$$
$$B^{\frac{1}{2}} = 100.$$

2. Take 1 with as many o's before it as there are Figures ſtanding before the laſt Period of A (or on the right Hand of it) call the Number ariſing B; then it is plain that B is a ſquare Number, whoſe Root is 1, with half as many o's as are in B. For to ſquare any Number expreſſed by 1 with a Number of o's before it, it's manifeſt, from the Nature of Multiplication, that the Square is 1, with double as many o's; wherefore B is a Square of as many Periods as A has, and being evidently contained in it, it follows, that the greateſt Square contained in it cannot have fewer.

C o r o l. The Root of the greateſt Square, contained in any Number A which is not a Square, hath as many Figures as A has Periods; for it has as many as its own Square has

has Periods (by *Corol. Lemma* 1.) which are as many as A has, by the prefent *Theorem*.

L E M M A III.

Any Number being diftributed into Periods, the greateft Square contained in the laft Period on the left, confidered as one Number by itfelf, is the Square of the laft Figure of the Root of the given Number, if it is a perfect Square; or of the Root of the greateft Square contained in it, if it's not a Square. Again; the greateft Square contained in the two laft Periods, taken as one Number by themfelves, is the Square of the laft two Figures of the Root of the given Number, or of the greateft Square contained in it; and the fame thing is true, comparing the 3 or 4, &c. laft Periods, with the Square of the 3 or 4, &c. laft Figures of the Root of the given Number, or of the greateft Square contained in it.

Demonftr. Let A be any Number, and B the Square Root thereof, or of the greateft Square contained in it; alfo let D reprefent the laft, or 2 laft, &c. Periods of A, [as in the annex'd Example, take D $=$ 22 or 2273, or 227338] and let r reprefent the laft, or 2 laft. &c. Figures of the Root B, [as here 4 or 47, or 476]; fo that r^2 is the Square of that laft, or 2, &c. laft Figures of the Root B.

Thefe things being fettled, the Truths to be proved are comprehended in one univerfal Cafe, which is this, *viz.* that r^2 in the greateft Square contained in D; which I fhall demonftrate in two Articles: Thus,

$A = 22733824.$ B4768
D r

1. r^2 is contained in D; for fince (by *Corol.* to *Lem.* 1 and 2.) there are as many Periods in A, as there are Figures in B; confequently there are, in every Cafe, as many Periods ftanding before D in the total A, as there are Figures before r in the total B; fo that taking D and r in their compleat Values, as they ftand in their Totals, there will be as many o's before D, as the Number of Figures in the Periods of A, which ftand before, or on the right Hand of D; and as many o's before r, as the half of thofe before D. [*Exam.* If D $=$ 22000000, then is $r =$ 4000; and if D $=$ 22730000, then is $r =$ 4700.] Then r^2 will have as many o's before it in its compleat Value, as double the Number of o's before the Root r in its compleat Value; and confequently as many as before D. We fhall now exprefs thefe Numbers in their compleat Values; thus, $r$oo, &c. r^2oo,oo, &c. Doooo, &c. and fhew that r^2 is contained in D. For,

If r^2 is greater than D, (both taken without the o's) then is r^2oooo, &c. greater than Doooo, &c. (r^2 and D being here equally multiplied) by an equal Number of o's prefix'd.) But A is equal to D, with as many Figures before it as there are o's before D or r^2 taken in their compleat Values; (i. e. Doooo, &c. r^2oooo, &c.) Therefore r^2oooo, &c. is greater than A; [for any Figures whatever in the Places of the o's before D, cannot be equal to the Excefs of r^2 above D, tho' that Excefs were but 1] i. e. the Square of $r$oo, &c. which is but a Part of B, is greater than the Square of B; becaufe A is at leaft equal to B² : but this is abfurd; therefore r^2 cannot be greater than D, and confequently muft be contained in it.

2. r^2 is the greateft Square contained in D: For fuppofe N² a greater Number than r^2; then take N with as many o's before it, as are before r in its compleat Value, and exprefs it thus, Noo, &c. fo that its Square is Noooo, &c. having double as many o's as Noo, &c. the Root has; or as many as r^2oooo, &c. or Doooo, &c. has. Now becaufe D contains N², by Suppofition; therefore Doooo, &c. contains N²oooo, &c. Alfo becaufe N² is fuppofed greater than r^2; therefore N is greater than r; and Noo, &c. greater than $r$oo, &c or than r with as many of any Figures before it; i. e. Noo, &c. is greater than B, (which is equal to r, with as many certain other Figures before it, as there are o's before

fore

fore r in its compleat Value $r$oo, &c. or in Noo, &c.) fo that Doooo, &c. a Part of A, contains N^2oooo, &c. the Square of Noo, &c. a Number which is greater than B, the Root of the greateft Square contained in A, which is abfurd; therefore r^2 is the greateft Square contained in D.

COROL. If we find the Root of the greateft Square contained in the laft Period of any Number, we have the laft Figure of the Root fought: And if we find the Root of the greateft Square contained in the two laft Periods, we have the two laft Figures of the Root fought, and fo on; which fo far explains the Inveftigation of the Rule; what remains to compleat it, you have in the following

LEMMA IV.

Part 1. If the Root of any known Square is fuppofed to confift of two Parts, or Members; then if one of thefe Members is known, we have a Rule for finding the other from the Confideration of the Square of a Binomial Root. Thus: If the Root is $A+B$, the Square is $A^2+2AB+B^2$, *viz.* the Sum of the Squares of the two Parts, and twice the Product of thefe Parts; wherein it is evident, that if the Square of either Part, as A^2, is fubtracted from the total Square $A^2+2AB+B2$, the Remainder is the Sum of the Square of the other Member, and the double Product of the two Members, *viz.* $2AB+B^2$. Now fuppofe A to be known; if we take 2 A for a Divifor, and find how oft it is contained in that Remainder; but under this Limitation, *viz.* that the Quote being added to the Divifor, and the Sum multiplied by the Quote, the Product fhall be equal to the Dividend $2AB+B^2$; then it is manifeft, that the Quote can be no other Number than B, the other Member of the Root fought. For fince $\overline{2A+B}\times B=2AB+B^2$ the Dividend, therefore it's plain that no other Number but B added to 2 A, and the Sum multiplied by the fame B, will produce $2AB+B^2$; fince either a greater or leffer Number added to 2A, makes a greater or leffer Sum; which being multiplied by the fame Number, produces ftill a greater or leffer Number.

Part 2. Tho' a Number is not a Square, yet having one Member of the Root of the greateft Square contained in it, we can find the other Member by the fame Method, as if it were a Square. Thus:

$M = A^2+2AB+B^2+R$

$M - A^2 = 2AB+B^2+R = D$

Let M be any Number not a Square, and $A+B$ the Root of the greateft Square contained in it; the Square is therefore $A^2+2AB+B^2$. Alfo let R be the Number that's more than $\overline{A+B}^2$ in M,

fo that $M = A^2+2AB+B^2+R$. Now A being known, if we take A^2 from M, the Remainder is plainly $2AB+B^2+R$, which we may call D. And if we find how oft 2 A is contained in D under this Limitation, *viz.* that the Quote being added to the Divifor, and the Sum multiplied by the fame Quote, the Product fhall ftill be lefs than D: [For this is to be obferved, that there is no Number which will make a Product equal to D; becaufe then M would be a Square; therefore any Number you can take, will make the Product either greater or leffer than D.] Then, I fay, the Quote is the other Member of the Root fought, *viz.* B: For let us fuppofe the Quote is another Number N, then if N is lefs than B, it follows, contrary to Suppofition, that N is not the greateft Number qualified according to the Rule, *viz.* which added to the Divifor, and the Sum multiplied by the fame Number, makes a Product lefs than D; for B is greater than N, and yet is a Number fo qualified, becaufe $D = 2AB+B^2+R = \overline{2A+B}\times B+R$: Therefore N is not lefs than B. Nor, again, can it be greater; for by Suppofition $\overline{2A+N}\times N$ ($= 2AN+N^2$) is lefs than D ($= M-A^2$) and adding A^2 to both, then $A^2+2AN+N^2$ ($= \overline{A+N}^2$) is lefs than M, and is therefore contained in it. But again; fince N is greater than

than B, A + N is greater than A + B, and $\overline{A + N}^2$ ($= A^2 + 2AN + N^2$) greater than $\overline{A + B}^2$ ($= A^2 + 2AB + B^2$) and confequently this is not the greateft Square contained in M, as was fuppofed: Wherefore N is not greater than B; and if it's neither greater nor leffer, it muft be equal.

COROLL. If a given Number M is not a Square, the Number R which is over the greateft Square contained in it, (and is neceffarily the Remainder, which happens in the Operation after B the fecond Member of the Root is found) may be greater than 2 A the Divifor; becaufe we have not taken 2 A out of the Dividend D as oft as poffible; but it can never exceed double of the Root found, if that is the true Root of the greateft Square contained in M: For let the Root found be called N, if the Remainder exceeds 2 N, it muft be at leaft 2 N + 1, and if this is added to N^2, the Sum $N^2 + 2N + 1$, ($= \overline{N + 1}^2$) being evidently contained in M, it follows that N is not the Root of the greateft Square contained in it, as was fuppofed.

APPLICATION *of the preceding* Lemma's *for demonftrating the* Extraction *of the* Square Root.

1. The firft and fecond *Lemma's* are already applied; from whence are deduced, as *Corollaries*, the firft Thing afferted in the *Rule*, viz. That the Root muft have as many Figures as the given Number has Periods.

2. From *Lemma* III. we have the Reafon why the given Number is pointed from the Right Hand to the Left; becaufe, being done fo, it is demonftrated that the laft Figure of the Root fought, the two laft, and fo on, make the Root of the greateft Square contained in the laft, the two laft, &c. Periods of the given Number.

3. The remaining Part of the Rule is to find the Figures of the Root, one after another, out of thefe Periods; the Reafon of which is contained in *Lemma* III. and IV. and its *Coroll.* and is deduced thus:

We firft take the laft Period, and the greateft Square contained in it we feek in the Table of fimple Squares, [which muft be found there; for fince a Period has but two Figures at moft, the Root of the greateft Square contained in it can be but one Figure; becauufe the Square of 10, the leaft Number of two Figures, is 100, which has three Figures]. The Root of this Square is, by *Lem.* 3. the laft Figure of the Root fought. So in the preceding *Example* 3. the given Number is 151426; the laft Period is 15, and the greateft Square contained in it is 9, whofe Root is 3, the laft Figure of the Root fought.

Now if we fuppofe the two laft Periods 1514 to be the given Number, then the Root of the greateft Square contained in it has but two Figures, whereof we have found the laft, viz. 3, whofe Real Value is 30; and to find the other, it's plain, from *Lemma* IV. that calling the firft Member of the Root now found, viz. $30 = a$, and calling the Member fought b, then the greateft Square contained in 1514 is $a^2 + 2ab + b^2$; but $a^2 = 9$, or rather 900 taken in its true Value; fo that 9 from 15, and 14 prefix'd to the Remainder, (which is the Method of the Rule) is the fame thing as 900 from 1514: The Remainder is 614, the fecond Refolvend, which is equal to $2ab + b^2$ at leaft, with fome Remainder over perhaps; we fhall therefore call the Remainder $2ab + b^2 + r$: What remains then, is to find this fecond Member of the Root b; and according to *Lem.* IV. if we make 22 the Divifor, and find how oft it is contained in the Refolvend $2ab + b^2 + r$, fo that calling the Quote b, this Quote added to the Divifor, and the Sum ($2a + b$) multiplied by b, the Product ($2ab + b^2$) fhall not exceed the Refolvend ($2ab + b^2 + r$); then it's fhewn that the Quote is truly the fecond Member of the Root: But it's manifeft that this is the very Method of the Rule; wherefore it's juft and

Z　　　　　　　　　　　　　　　　good

good when the Root fought has but two Figures. Again; The Number given having three Periods, as if it were 151426, then having found 38 the Root of the greateft Square contained in the two firft Periods 1514, (as already fhewn); thefe are the two laft Figures of the Root of 151426, (by *Lem.* 3.) And if we take 38 in its true Value it is 380, becaufe there is another Figure on its Right in the Root fought; then 380 being confidered as one Member of the Root fought, we call it alfo *a*; and by *Lemma* IV. we are to fubtract its Square, *viz.* $\overline{300+80}^2 = \overline{300}^2 + 2 \times 300 \times 80 + \overline{80}^2$ out of the given Number 151426: But this is already done, becaufe we have taken firft the Square of 3, (which was in the former Step called *a*) *viz.* 9, out of 15, which is equivalent to taking the Square of 300, (which is now *a*) *viz.* 90000, out of 151426, which leaves 61426; then *b* being 8, and *a* = 30, we have taken $2ab + b^2 = 544$ out of 614, the former Remainder; to the Remainder 70 we have prefix'd 26, the firft Period, which makes the whole 7026; and this is the fame Number which remains, if taking *b* = 80, and *a* = 300, we take $2ab + b^2 = 48000 + 6400 = 54400$ out of the former Remainder 61426. Now the Square of 380 being taken out of the given Number 151426, the Remainder 7026 is the next Refolvend; and for a Divifor we have made $2 \times a = 2 \times 380 = 760$, and the Member fought we have found the fame way as before, which is both according to *Lemma* IV. and the Rule for Extraction; which is therefore good for any Root of three Figures.

If there are more than three Figures in any Root, the Reafons of the Rule from one Step to another for ever are manifeftly the fame, and need not be further infifted on. I fhall only illuftrate this Application by one Example of a perfect Square, whofe Involution by the Method fhewn in the preceding Section, and its Evolution by the prefent Rule, will illuftrate one another; and you'll evidently perceive, that as by knowing the true Place of every Figure found in the Root, we may take it in its compleat Value, and perform the Work that way, as in the following Operation; yet we fave the trouble of many fuperfluous Figures by the Method of the Rule, which produces the fame Effect.

Involution *of* 389 *to its Square makes* 151321.
Thus:

Root.
$389 = 300 + 80 + 9$

$90000 = a^2.\ (a = 300)$
$48000 = 2ab\ (b = 80)$
$6400 = b^2$

$144400 = \overline{a+b}^2 = 380^2 = A^2$
$6840 = 2AB,\ (B = 9)$
$81 = B^2$

$151321 = \overline{A+B}^2 = \overline{380+9}^2 = \overline{389}^2$

Evolution *of* 151321 *to its Square Root makes* 389.
Thus:

$$\overset{A \qquad\qquad B}{\overset{\frown\frown}{}}$$

Square.   *a*   *b*
$151321\ (300 + 80 + 9$
$a^2 = 90000\ ($

Divifor. $2 \times 300 \overset{a}{\big)}\ \overline{61321}$ Refolvend.

$48000 = 2ab$
$6400 = b^2$

$54400 = 2ab + b^2$

Divifor. $2 \times 380 \overset{A}{\big)}\ \overline{6921}$ Refolvend.

$6840 = AB$
$81 = B^2$

$6921 = 2AB + B^2$

0000

Thefe

Thefe Operations are reverfe of one another; and as to the Evolution, it differs from the Method of the Rule in this only, that the feveral Members of the Root are here written in their compleat Value, which occafions the writing down many Figures unneceffarily, which we avoid the other Way.

There remain yet two things to be demonftrated, which are delivered in *Schol.* 1. and 2.

1*ft.* The Remainder, after every Figure of the Root is found, cannot exceed the Sum of the Divifor and double the Quote: The *Reafon* of this is contained in *Corol. Lemma* IV. where it's fhewn, that what's over the greateft Square contained in any Number cannot exceed double the Root of the greateft Square; which is plainly the Sum of the Divifor at every Step, and double the Quote; for the Divifor is double of all the preceding Figures taken in their compleat Value, which therefore added to double the Quote, makes double all the Root. Thus, if any Root is $a + b$, the Divifor for finding b is $2a$; and when $2ab + b^2$ is taken out of the Refolvend, call the Remainder r; and in the *Coroll.* to *Lemma* IV. it's fhewn that r cannot exceed $2a + 2b$.

2. If the Divifor is contained oftner than 9 times in the Refolvend, after the fecond Step, or after the fecond Figure of the Root is found, the Figure fought is 9. For fince the Refolvend contains the Divifor ($2a$) at leaft 10 times, it may be reprefented thus; $2a \times 9 + 2a + R$. Now taking 9 for the Quote, the Product according to the Rule is $2a \times 9 + 9 \times 9$, which cannot exceed the Refolvend, becaufe 9×9 cannot exceed $2a$, which in this Cafe exceeds 100, fince there being two Figures found in the Root, and 0 prefix'd to them in order to form the Divifor $2a$, then is $2a$ a Number of at leaft three Places, which is therefore greater than $9 \times 9 = 81$. Laftly, fince it is demonftrated that in every Step, the Quote, under the Limitation of the Rule, is but one Figure; and 9, which is the greateft Number of one Figure, makes a Product not exceeding the Refolvend; therefore 9 is the Number fought.

In the fecond Step, if the Divifor is oftner than 9 times contained in the Refolvend, then it's plain, that if 81 is lefs than $2a$, [as it will certainly be when a is 5, 6, 7, 8, or 9, i.e. 50, 60, 70, 80, or 90, as they are taken in forming the Divifor; for then the Doubles, or $2a$, are 100, 120, 140, 160, 180.] then 9 is the Figure fought; becaufe $2a \times 9 + 81$ muft be lefs than $2a \times 9 + 2a + R$, the Refolvend, fince 81 is lefs than $2a$. But if the firft Figure is 1, 2, 3, or 4, that is, if $2a$ is 2×10, 2×20, 2×30, or 2×40, i. e. 20, 40, 60, or 80, which are lefs than 81, then the Figure fought will be lefs than 9, if $2a + R$ is lefs than 81; and it will be 9, if $2a + R$ is equal to or greater than 81 : Becaufe the Refolvend being $2a \times 9 + 2a + R$, if the Figure fought is made 9, then the thing to be fubtracted from the Refolvend is $2a \times 9 + 81$, fo that $2a + R$ muft be at leaft equal to 81 ; and if it is not, we muft therefore take a lefs Figure for the Quote, fo that the Refolvend be at leaft equal to the Number to be fubtracted.

Of the Proof of the Square Root.

As Extraction is oppofite to the Raifing of Powers, fo the one is the Proof of the other : Thus; To prove the Square Root, multiply it by itfelf, and if the Product is equal to the given Square, or to the given Number after the Remainder of the Extraction is taken out of it, then the Extraction is right done.

But this may be alfo proved by cafting out of 9's: Thus; Caft the 9's out of the given Number, if there is no Remainder in the Extraction; or out of the Difference of that Number and Remainder; then caft the 9's out of the Root found, and multiply the Excefs (or what it wants of 9) by itfelf, and caft the 9's out of the Product; if the Excefs, or what it wants of 9, is equal to the preceding Excefs of 9's, the Extraction is right.

Example 1. The Square Root 256 is 16; proved thus; the Excefs of 9's in 256 is 4. In 16 it is 7, and this multiplied by itfelf is 49, in which the Excefs of 9's is alfo 4.

Example 2. The greateſt Square Root contained in 230 is 15, and 6 remains: For 230—6 = 225, in which there is 0 over 9's; then in 15 there is 6 over 9; and 6 × 6 = 36, in which there is alſo 0 over 9's.

The *Reaſon* of this Practice is evident from what is demonſtrated of it in Multiplication.

Problem II. *To Extract the* Cube Root *of a* Whole Number.

Roots.	Cubes.
1 :	1
2 :	8
3 :	27
4 :	64
5 :	125
6 :	216
7 :	343
8 :	512
9 :	729

R u l e I. MAKE a Table of ſimple Cubes, as in the Margin; then

II. Diſtribute the given Number into Periods of three Figures, beginning at the Right Hand: The Number of Periods ſhews the Number of Figures in the Root.

III. Begin at the laſt Period, which is the firſt Reſolvend, and ſeek it or the next Cube Number leſs than it in the Table of ſimple Cubes, the Root of that is the laſt Figure (or that in the higheſt Place) of the Root ſought; which being ſet down, ſubtract its Cube out of the laſt Period; to the Remainder prefix the next Period, and you have the next Reſolvend.

IV. Conſider the Figure found as in the Place of 10's, or with 0 prefix'd, and under that Value take the Triple of it, and alſo the Triple of its Square, making the Sum of theſe the Diviſor; [which being compoſed of two Parts, it will be convenient to diſtinguiſh them by calling the Triple Square the firſt Part, and the other the ſecond Part.] Then find how oft this Diviſor is contained in the Reſolvend laſt formed; which muſt never be taken above 9, (tho' it may be oftner contained) and then alſo it muſt be under this Limitation, *viz.* That having multiplied the firſt Part of the Diviſor by the Quote, (now found) and the ſecond Part by the Square of that Quote; and, laſtly, to the Sum of theſe two Products adding the Cube of the Figure found; this Sum ſhall not exceed the Reſolvend: which Sum being therefore ſubtracted out of the Reſolvend, and the next Period prefix'd to the Remainder, you have the next Reſolvend.

V. Take both the Figures of the Root already found, and conſidering them as ſo many 10's, *i. e.* place 0 before them, and under that Value take the Triple of that Number, and alſo the Triple of its Square; whoſe Sum is your next Diviſor, diſtinguiſhed into firſt and ſecond Part, as before: Then find how oft this Diviſor is contained in the Reſolvend laſt formed, under the ſame Limitations as before; place the Figure found on the Right of theſe already found in the Root; and ſubtracting from the Reſolvend, as formerly directed, to the Remainder prefix the next Period for a new Reſolvend.

VI. Make a new Diviſor from the Figures of the Root found in the ſame manner as in the preceding Steps, and divide, under the ſame Limitation; and thus proceed till all the Periods are taken in; finding at every Step a new Figure of the Root: which will in ſome Caſes be 0, as when the Diviſor is greater than the Reſolvend, or when 1 added to the Diviſor makes the Sum greater than the Reſolvend; in which Caſe, after the 0 is ſet in the Root, prefix the next Period to the ſame Reſolvend; and go on, forming a new Diviſor by prefixing another 0 to the laſt Diviſor.

S C H O-

S C H O L I U M S.

Thefe *particular* O B S E R V A T I O N S may be ufefully added to this Rule (tho' it's a compleat *general* Rule by itfelf):

1. If you begin your Trials for the Quote at the greateft Number of Times (not exceeding 9) that the Divifor is contained in the Refolvend; then the Limitation of the Rule, for the Number to be compared with the Refolvend, is fufficient to determine when you have the true Figure. Yet it will be ufeful to *obferve*, That if you fhould begin at a leffer Figure than the Remainder, tho' it may be greater than the Divifor, yet it muft never exceed the Sum of thefe two Numbers, *viz.* Triple all the Figures of the Root already found, (taken as one Number) and triple its Square.

2. If the Divifor is contained in the Refolvend oftner than 9 times, at the fecond Step, or when you feek the fecond Figure of the Root; and if, at the fame time, the firft Figure is 8 or 9, then the Figure fought is certainly 9: But if the firft Figure is lefs than 8, you muft make Trials. Again; If in any Step after the fecond, the Divifor is oftner than 9 times contained in the Refolvend, the Figure fought is certainly 9.

3. If there is a Remainder after all the Periods are employ'd, the given Number is not a Cube; and the Root found is that of the greateft integral Cube contained in it. How to find a Mixt Root whofe Cube fhall be within any affigned Difference from the given Number, fhall be taught in its proper Place. —— ——

E X A M P L E S.

Example 1. The Cube Root of 614125 is 85.

Operation.

$$614125 \quad (85 \quad (a.b$$
$$a^3 = 512$$

$$3a^2 + 3a = 19440) \quad 102125. \text{ 2d Refolvend.}$$
(wherein $a = 80$)
($a^2 = 6400$)

$$96000 = 3a^2 \times b$$
$$6000 = 3a \times b^2$$
$$125 = b^3$$

$$102125. \text{ Sum.}$$

$$000000. \text{ Remainder.}$$

Explication for Exam. 1.

The given Number being pointed, the laft Period is 614, and the next Cube to that is 512, whofe Root is 8, which I call *a*, and fubtracting $a^3 = 512$ from 614, the Remainder is 102, to which the next Period prefix'd makes 102125, the 2d Refolv. Then for a Divifor I take $a = 80$, and fo $a^2 = 6400$, and $3a^2 = 19200$; then $3a = 240$, and $3a^2 + 3a = 19200 + 240 = 19440$, the Divifor; which is contained in the Refolvend, under the Limitation of the Rule, 5 times, the 2d Figure of the Root, which calling *b*, then $3a^2 \times b = 96000$, $3a \times b^2 = 6000$, $b^3 = 125$, and the Sum of thefe is 102125, equal to the Refolvend; fo that the given Number is a true Cube, whofe Root is 85.

Example

Example 2. The Cube Root of 41421736 is 346.

Operation.

$$41421736 \quad \binom{a\,b}{346}$$
$$a^3 = 27 \qquad\qquad \overline{A\,B}$$

$$3\,a^2 + 3\,a = 2790) \quad 14421.\ \text{2d Refolvend.}$$
$$(a = 30)$$
$$(a^2 = 900)$$

$$10800 = 3\,a^2 \times b$$
$$1440 = 3\,a \times b^2$$
$$64 = b^3$$

$$12304\ \text{Sum.}$$

$$3\,A^2 + 3\,A = 347820) \quad 2117736.\ \text{3d Refolvend.}$$
$$(A = 340)$$
$$(A^2 = 115600)$$

$$2080800 = 3\,A^2 \times B$$
$$36720 = 3\,A \times B^2$$
$$216 = B^3$$

$$2117736\ \text{Sum.}$$

$$0000000\ \text{Remainder.}$$

Explication for *Exam.* 2.

The firſt Period is 41, and the next Cube is 27, whoſe Root is $3 = a$. The ſecond Refolvend is 14421: Then for the Diviſor I take $a = 30$, whereby $3\,a^2 = 2700$, $3\,a = 90$, and $3\,a^2 + 3\,a = 2790$, the Diviſor, by which the ſecond Figure of the Root is 4; which is proved by the *Operation*: for $3\,a^2b + 3\,ab^2 + b^3 = 12304$, (as in the Work) which ſubtracted from the Refolvend leaves 2117; which is a Number not exceeding three times 34. $+$ 3 × Square of 34, as *Schol.* 1. preſcribes; for 34 × 34 = 115600, whoſe Triple is 346800, and 3 × 34 = 102, then $346800 + 102 = 346902$. Or ſhould we have begun to gueſs for the Figure ſought, the greateſt Quote is 5, which is too great for the Limitation of the Rule, and 4 the next Number not being too great, muſt be the true Number. The third Refolvend is 2117736; and to form the Diviſor, I take $A = 340$, whence $A^2 = 115600$; and the Diviſor is 347820, whence the laſt Figure of the Root is 6, as the Work ſhews.

Example 3. If we ſeek the Cube Root of 705919947284, we find it's not a perfect Cube, but the Root of the greateſt Cube contained in it is 8904; whoſe Cube is 705919947264, and the Remainder is 20, as you ſee in the Operation.

Operation.

$$705919947284 \quad (8904$$
$$8^3 = 512$$

$$\text{Diviſor} = 3\,a^2 + 3\,a = 19440) \quad 193919.\ \text{2d Refolvend.}$$
$$(a = 80,\ a^2 = 6400)$$

$$172800 = 3\,a^2\,b$$
$$19440 = 3\,a\,b^2$$
$$729 = b^3$$

$$192969\ \text{Sum.}$$

$$(a = 890)\ 3\,a^2 + 3\,a = 2378970 \}\quad 950947.\ \text{3d Refolvend.}$$
$$(a = 8900)\ 3\,a^2 + 3\,a = 237656700 \}\quad 950147284.\ \text{4th Refolvend.}$$

$$950520000 = 3\,a^2\,b$$
$$387200 = 3\,a\,b^2$$
$$64 = b^3$$

$$950907264.\ \text{Sum.}$$

$$20.\ \text{Remainder.}$$

Here the 3d Refolvend being leſs than the Diviſor, I put o in the Root, and form a new Refolvend by the next Period. The reſt of the Work is obvious.

DE-

DEMONSTRATION *of the preceding Rule.*

L E M M A I.

If any three Numbers are multiplied into one another, the Product can have at moſt but as many Figures as are in all the three Factors, and at leaſt but two fewer. Example: $3 \times 4 \times 9 = 108$.

Demonſtr. This is a plain Conſequence of *Lem.* I. for the Square Root: Becauſe the Product of two Factors cannot have more Places than are in both Factors, or but one fewer at leaſt ; which Product being conſidered as one Factor, and multiplied by a third Factor, the ſame is true of this new Product ; which makes the Truth propoſed manifeſt.

Coroll. The Cube of any Number can have at moſt but as many Figures as triple the number of Figures in the Root, and but two fewer at leaſt. Wherefore, again ; Any Cube being diſtributed into Periods of three Places, the Number of Periods, and the Number of Figures in the Root muſt neceſſarily be equal ; and the laſt Period may, in ſome Caſes, conſiſt only of one or two Figures.

L E M M A II.

If any Number A is not a Cube, yet being diſtributed into Periods, according to the preceding Rule, the greateſt Cube contained in it, as N^3, will have preciſely as many Periods as that Number A has.

Example: 35987 is not a Cube ; but being pointed has two Periods, *viz.* as many as the greateſt Cube contained in it, 35937.

$A = 35987$
$N^3 = 35937$

$B = 1000$
$B^{\frac{1}{3}} = 10$

Demonſtration 1. N^3 cannot have more Periods than A, for then it will have more Figures, and conſequently be a greater Number than A, contrary to Suppoſition.

2. Take 1 with as many o's before it as there are Figures ſtanding before the laſt Period of A ; (*i. e.* on the right Hand of it) call the Number ariſing B : Then it's plain that is a Cube Number, whoſe Root is 1, with as many o's before it as ⅓ Part of the Number of o's before the 1 in the Cube B ; for to cube any Number expreſſed by 1 with o's, it's manifeſt from the Nature of Multiplication, that the Cube is 1 with three times as many o's ; wherefore B is a Cube of as many Periods as A has ; and being evidently contained in it, it follows, that the greateſt Cube contained in it cannot have fewer.

Coroll. The Root of the greateſt Cube contained in any Number A, which is not a perfect Cube, hath as many Figures as A hath Periods : For it hath as many as its own Cube hath Periods, (by *Coroll. Lemma* I.) which are as many as A has, by the preſent *Theorem.*

L E M M A III.

Any Number being diſtributed into Periods according to the Rule, the greateſt Cube contained in the laſt Period on the left, conſider'd as one Number by itſelf, is the Cube of the laſt Figure of the Root of the given Number, if it's a perfect Cube ; or of the Root of the greateſt Cube contained in it, if it's not a Cube. Again ; the greateſt Cube contained in the two laſt Periods, taken as one Number by themſelves, is the Cube of the laſt two Figures of the Root of the given Number, or of the greateſt Cube contained in it. And the ſame thing is true, comparing the 3 laſt or 4 laſt Periods, and ſo on, with the

2   Cube

Cube of the 3 or 4, &c. laſt Figures of the Root of the given Number, or of the greateſt Cube contained in it.

Demonſtr. Let A be any Number, and B the Cube Root thereof, or of the greateſt Cube contained in it. Alſo let D be the laſt, or 2 laſt, or 3 laſt, &c. Periods of A; [as in the annex'd Example, take D = 41, or 41421] and let r repreſent the laſt, or two laſt, or three laſt, &c. Figures of the Root B, (as here 3 or 34) ſo that r^3 is the Cube of that laſt or two laſt, &c. Figures of the Root. Theſe things being ſettled, the Truths to be proved are comprehended in one univerſal Caſe, thus; *viz.* r^3 is the greateſt Cube contained in D; which I ſhall demonſtrate in two Articles, thus:

$$A = 41421736 \brace D \qquad B = 346. \brace r$$

1. r^3 is contained in D: for ſince (by *Cor.* to *Lem.* 1, and 2.) there are as many Periods in A, as there are Figures in B; conſequently there are, in every Caſe, as many Periods ſtanding before D in the total A, as there are Figures before r in the total B; ſo that taking D and r in their compleat Value, as they ſtand in their Totals, there will be as many 0's before D, as the Number of Figures of the Periods in A, which ſtand before, or on the right Hand of D; and as many 0's before r, as the third Part of theſe before D. [For Example: If D = 41,000,000, then is r = 300. If D = 41421,000, then is r = 340.] Then r^3 will have as many 0's before it, in its compleat Value, as triple the Number of 0's before the Root r in its compleat Value, and conſequently as many as are before D. We ſhall now expreſs theſe Numbers in their compleat Values, thus, r00, &c. $r^3$000000, &c. D000,000, &c.

Again; If r^3 is greater than D, then is $r^3$000,000, &c. greater than D000,000, &c. (being equally multiplied by the equal Number of 0's prefix'd.) But A is equal to D, with as many certain Figures before it as there are 0's before D, or r^3 in their compleat Values, *i. e.* D000,000, &c. or r000,000, &c. therefore $r^3$000,000, &c. is greater than A; [for any Figures whatever in the Places of the 0's before D, cannot be equal to the Exceſs of r^3 above D, tho' that Exceſs were but 1]; *i. e.* the Cube of r00, &c. which is but a Part of B, is greater than the Cube of B, becauſe A is at leaſt equal to B³. But this is abſurd; therefore r^3 cannot be greater than D, and conſequently muſt be contained in it.

2. r^3 is the greateſt Cube contained in D. For ſuppoſe N³ a greater Number than r^3; then take N with as many 0's before it, as are before r in its compleat Value, and expreſs it thus, N00, &c. ſo that its Cube is N000,000, &c. having triple as many 0's as N00, &c. the Root has; or as many as $r^3$000,000, &c. or D000,000, &c. has. Now becauſe D contains N³ (by Suppoſition) therefore D000,000, &c. contains N³000,000, &c. Alſo, becauſe N³ is ſuppoſed greater than r^3, therefore N is greater than r, and N00, &c. greater than r00, &c. or than r with as many of any Figures before it; *i. e.* N00, &c. is greater than B, (which is equal to r with as many certain other Figures before it, as there are 0's before r in its compleat Value r00, &c. or in N00, &c.) ſo that D000,000, &c. a Part of A contains N³000,000, &c. the Cube of N00, &c. a Number which is greater than B, the Root of the greateſt Cube contained in A, which is abſurd; therefore r^3 is the greateſt Cube contained in D.

COROL. If we find the Root of the greateſt Cube contained in the laſt Period of any Number, we have the laſt Figure of the Root ſought; and if we find the Root of the greateſt Cube contained in the two laſt Periods, we have the two laſt Figures of the Root ſought, and ſo on. Which ſo far explains the Inveſtigation of the Rule; what remains to compleat it, you have in the following

LEMMA IV.

Part I. If the Root of any known Cube is ſuppoſed to conſiſt of two Parts; then if one of theſe Parts is known, we can find the other by means of the Cube of a Binomial Root. Thus: $A + B$

$\overline{A+B}^3 = A^3 + 3A^2B + 3AB^2 + B^3$; wherein it's evident, that if the Cube of the known Part A, *viz.* A^3, is subtracted from the Cube of the whole, the Remainder is $3A^2B + 3AB^2 + B^3$. Now since A is known, so also is $3A^2 + 3A$; and if we seek how oft this is contained in the preceding Remainder, under this *Limitation*, that the first Member of the Divisor, $3A^2$, being multiplied by the Quote, and the second Member $3A$ being multiplied by the Square of the Quote, and to these two Products the Cube of the Quote be added, the Sum shall be equal to the Dividend; then the Quote shall be equal to B the Number sought; because no other Number but B can answer to this Condition: For if you call the Quote D, then must $3A^2D + 3AD^2 + D^3$ be equal to $3A^2B + 3AB^2 + B^3$; which is manifestly impossible, unless D be $=B$; since otherwise the respective Members of the one will be lesser or greater than those of the other, and consequently the Wholes will not be equal.

Part 2. Tho' a Number is not a Cube, yet having one Member of the Root of the greatest Cube contained in it, we can find the other by the same Method, as if it were a Cube. Which will easily appear. *Thus* :

$M = A^3 + 3A^2B + 3AB^2 + B^3 + R$

$M - A^3 = 3A^2B + 3AB^2 + B^3 + R = D.$

Let M be a Number, not a Cube; and $A+B$ the Root of the greatest Cube contained in it; which Cube is therefore $A^3 + 3A^2B + 3AB^2 + B^3$. Again; let R be the Number that's more than that Cube in M; so that $M = A^3 + 3A^2B + 3AB^2 + B^3 + R$. Now A being known, take A^3 from M, the Remainder is $3A^2B + 3AB^2 + B^3 + R$; which we may call D: And then if we find how oft $3A^2 + 3A$ is contained in D, under these Limitations, *viz.* that the Quote being multiplied into $3A^2$, and the Square of the Quote multiplied into $3A$, and to these Products the Cube of the Quote be added, the Sum shall still be less than D. [For *observe*, that whatever Number you chuse for the Quote, it will make this Sum either greater or lesser than D, and never equal; because were it equal, then M would be a Cube, contrary to Supposition.] Then, I say, the Quote is equal to B, the other Member of the Root sought. Because, if it can be different, suppose it to be N ; which is either lesser or greater than D: But it cannot be lesser; for then it would follow, that, contrary to Supposition, N is not the greatest Number qualified according to the Rule, *viz.* so that $3A^2N + 3AN^2 + N^3$ is less than D ; for B is greater than N, and yet is so qualified, since $D = 3A^2B + 3AB^2 + B^3 + R$: Wherefore N cannot be less than B ; nor can it be greater, because, by Supposition, $3A^2N + 3AN^2 + N^3$ is less than D $(= M - A^3)$; and adding A^3 to both, then $A^3 + 3AN^2 + 3A^2N + N^3$ $(= \overline{A+N}^3)$ is less than M, and therefore is contained in it. But again ; $A+N$ is greater than $A+B$, and $\overline{A+N}^3$ greater than $\overline{A+B}^3$, consequently $\overline{A+B}^3$ is not the greatest Cube contained in M, contrary to Supposition ; so that N cannot be greater than B : Wherefore, lastly, since N cannot be either lesser or greater than B, it must be equal to it.

C o r o l. If a Number M is not a Cube, the Number R, which is over the greatest Cube contained in it, (which is necessarily the Remainder after the second Member B is found) can never exceed the Sum of triple the Root found, and triple its Square : For if the Root found is N, then if the Remainder exceed $3N^2 + 3N$, it must be at least $3N^2 + 3N + 1$; which added to N^3 makes $N^3 + 3N^2 + 3N + 1 = \overline{N+1}^3$. And since this Cube is manifestly contained in M, (for it's the Sum of the greatest Cube N^3 contained in M, and the Remainder $3N^2 + 3N + 1$ added); it follows, contrary to Supposition, that N is not the Root of the greatest Cube contained in M, because $N+1$ is greater than N ; and $\overline{N+1}^3$ is contained in M, if R is greater than $3N^2 + 3N$; therefore this cannot be.

A a

APPLICATION *of the preceding* Lemma's *for demonstrating the Extracting of the* Cube Root.

1. The firſt and ſecond *Lemma's* are already applied; from whence are deduced as *Corollaries*, the firſt thing aſſerted in the Rule, *viz.* That the Root muſt have as many Figures as the given Number has Periods.

2. From *Lem.* 3. we have the Reaſon why the given Number is pointed from the right Hand, *viz.* becauſe, being done ſo, it is demonſtrated, that the laſt Figure of the Root ſought, (*i. e.* the Figure in the higheſt Place) the two laſt, *&c.* make the Root of the greateſt Cube contained in the laſt, or two laſt, *&c.* Periods of the given Number.

3. The remaining Part of the Rule is to find the Figures of the Root, one after another, out of theſe Periods; the Reaſon of which is contained in *Lem.* 3 and 4, and its *Corol.* and is deduced thus:

We firſt take the laſt Period; and in the Table of ſimple Cubes, we ſeek that Number, or the next Letter, whoſe Root is, by *Lem.* 3. the higheſt Figure of the Root ſought. So in the preceding *Example*, the given Number is 41421736, which we ſhall here call N. The laſt Period is 41, and the next Cube to this is 27, whoſe Root is 3, the laſt Figure of the Root ſought. Now if we ſuppoſe the two laſt Periods 41421 to be the whole of the given Number, then the Root of the greateſt Cube contained in it has but two Figures, whereof we have now found the laſt; and to find the other (which is the next Figure of the Root ſought, by *Lem.* 3.) we proceed thus: Calling the Figure found *a*, we ſubtract its Cube $a^3 = 27$, from 41 the laſt Period; and to the Remainder 14 prefixing the next Period 421, the whole 14421 is the 2d Reſolvend. And obſerve, that as the 41 is really 41000, in reſpect of the total 41421; ſo the Figure found is really 30, in reſpect of the next to be found; and in that Value we do actually take it by ſubtracting it from 41, conſidering where this ſtands, and which the prefixing the next Period to the Remainder does farther clear: For this is the ſame thing as if we had written 27000 the Cube of 30, and taken that from 41421; wherefore this is the ſame Operation as that explained in *Lem.* 4. *i. e.* having found 30 the firſt Member of the Root of 41421, we take its Cube out of the whole; and out of the Remainder 14421, we ſeek the next Member of the Root, which we know cannot exceed 9, becauſe it's the ſecond Member of a Root conſiſting of two Figures; whereof we have found that belonging to the higheſt Place, which conſider'd in its compleat Value is the firſt Member. Now to find the Figure ſought, we form a Diviſor according to the Rule (demonſtrated in *Lem.* 4.) thus: Taking $30 = a$, the Diviſor is $3 a^2 + 3 a = 3 \times 900 + 3 \times 30 = 2700 + 90 = 2790$. And this we find contained in the Reſolvend 14421, 5 times; but under the Limitation of the Rule we can take it at moſt 4 times; and 4 is the Figure ſought; which calling *b*, the Proof of its being the true Figure is this: We take $3 a^2 b + 3 a b^2 + b^3 = 12304$, which is leſs than the Reſolvend 14421; and 4 is therefore the right Figure, becauſe 5 would have made $3 a^2 b + 3 a b^2 + b^3$ greater than 14421. Or had we at a gueſs taken $3 = b$, then would it be $3 a^2 b + 3 a b^2 + b^3 = 8937$; which taken from 14421 leaves 5484; which is greater than $3 \times \overline{33}^2 + 3 \times 33 = 3 \times 1089 + 3 \times 33 = 3267 + 99 = 3366$; and therefore 3 is too little, as is ſhewn in *Schol.* 1. added to the Rule. Thus we have found 34, the Root of the greateſt Cube contained in 41421, (the Remainder, or what is over, being 2117) and have ſhewn that the Rule is juſt and good for a Root of two Figures. Again; For a Number of three Periods, as 41421736, whoſe Root has three Figures; having found the two Figures in the higheſt Places; and taking theſe with o prefix'd, which makes the true Value; and calling this again *a*, or A, the firſt Member of the Root, the ſecond Member, which is a ſingle Figure, is found the ſame way as before explained; which is according to the Rule. But now in this there is ſo much of the Work already done; for the Cube of this

this firſt Member, or A³, is already ſubtracted from the Total 41421736, becauſe A is now equal to the former $a + b$; and it's evident from the Work that we have ſubtracted a^3, and then $3a^2b + 3ab^2 + b^3$ to make the Cube of $\overline{a+b}$. It's true, we have taken $a = 30$ and $b = 4$; whereas A = 380, ſo that a ſhould be 300 and $b = 80$: But by the Places in which we have ſet a^3, and $3a^2b + 3ab^2 + b^3$, we have in effect taken them, as if it had been $a = 300$ and $b = 80$; and ſo we have duly ſubtracted A³, or the Cube of 380, from the Total 41421736; the Remainder whereof is 2117, to which the next Period 736 is prefix'd, making 2117736 the Reſolvend for finding the next Figure; which we find to be 6, by the ſame Rule and Reaſon as we found the laſt Figure.

If there are more than three Figures in any Root, the Reaſons of the Rule are manifeſtly the ſame from one Step to another *in infinitum*. I ſhall add for an Illuſtration one Example, wherein each Figure of the Root is taken in its compleat Value.

Involution *of* 346 *to its Cube, makes* 41421736.

Thus:

Root
$$346 = 300 + 40 + 6$$

$$27000000 = a^3. \ (a = 300.)$$
$$10800000 = 3a^2b. \ (b = 40.)$$
$$1440000 = 3b^2a.$$
$$64000 = b^3.$$

$$39304000 = \overline{a + b}^3 = A^3.$$
$$2080800 = 3A^2B. \ (A = 340. \ B = 6.)$$
$$36720 = 3AB^2.$$
$$216 = B^3.$$

$$41421736 = \overline{A + B}^3.$$

Evolution *of* 41421736 *to its Cube Root, makes* 346.

Thus:

$$
\begin{array}{c}
\quad\quad\quad\quad\overset{A\ \ \ \ B}{\overbrace{}} \\
\quad\quad\quad\quad (a \quad\ b \\
41421736(300 + 40 + 6. \\
a^3 = 27000000
\end{array}
$$

$$3a^2 + 3a = 270900) \ 14421736 \ \text{Reſolvend.}$$

$$10800000 = 3a^2b.$$
$$1440000 = 3ab^2.$$
$$64000 = b^3.$$

$$12304000 \ \text{Sum.}$$

$$3A^2 + 3A = 347820) \ 2117736 \ \text{Reſolvend.}$$

$$2080800 = 3A^2B.$$
$$36720 = 3AB^2.$$
$$216 = B^3.$$

$$2117736 \ \text{Sum.}$$

$$000000$$

There remain yet two things to be demonſtrated, which are deliver'd in *Schol.* 1 and 2. *viz.*

1. The Remainder can never exceed the Sum of theſe two Numbers, *viz.* triple all the Figures of the Root already found (taken as one Number) and triple the Square of the ſame; the Reaſon of which you have plainly in *Cor. Lem.* 4.

2. If the Diviſor $3a^2 + 3a$ is contained oftner than 9 times in the Reſolvend, then if it's ſo after the ſecond Figure is found, 9 is the Figure ſought; for in this Caſe the Reſolvend may be thus repreſented, $3a^2 \times 10 + 3a \times 10 + R = 3a^2 9 + 3a^2 + 3a \times 10 + R$, and the Sum upon which the Limitation depends, being $3a^2b + 3ab^2 + b^3$, if b is 9, then this Sum is $3a^2 \times 9 + 3a \times 81 + 729$: Compare this with the Reſolvend, they have this Part in common, *viz.* $3a^2 \times 9$: Set this aſide, and compare the Remainders in both, *viz.* $3a^2 + 3a \times 10 + R$, and $3a \times 81 + 729$; this laſt is leſs than the former; for, after two Figures of the Root are found, $\bar{a}$ conſiſts of three Figures, in its compleat Value, and ſo muſt be at leaſt 100: Therefore $3a \times 81$ is leſs than $3a^2$, and 729 is leſs than $3a \times 10$, which is at leaſt $300 \times 10 = 3000$. Hence it is plain, that the Reſolvend is greater than the Number to be compared with it in the Limitation of the Quote; and the greater that

$\bar{a}$ is,

a is, as it will be always greater at every Step after the fecond, fo much will the Refolvend exceed that other Number.

Again; In the fecond Step, the Quote is certainly 9, if the firft Figure found is either 8 or 9; *i. e.* if $A = 80$ or 90; which you'll find by comparing as before $3 A^2 + 3 A \times 10 + R$ with $3 a \times 81 + 729$; for putting $a = 80$ or 90, you'll find $3 A \times 81 + 729$ lefs than $3 A^2 + 3 A \times 10$: But when A is fuppofed 70, or 60, *&c.* it will be greater, and therefore the Quote muft be lefs than 9, unlefs the Number R, which belongs to the Refolvend, is greater than the Excefs of $3 A \times 81 + 729$ above $3 A^2 + 3 A \times 10$, as in fome Cafes it will, and in fome it will not.

S C H O L I U M, *concerning a different Method of Practice in the Extraction of a* Cube Root.

The preceding Rule is nearly according to the moft common Method, that it might be accommodated to the Principles from which the Reafon and Demonftration of it might be moft eafily deduced: But there is another Method, differing a little in one of the principal Steps, which is this:

Having pointed the given Number, and found the firft Figure of the Root; then in all the fucceeding Steps form the Divifor as before, and find the Quote under this Limitation, *viz.* That being added to the Divifor, and the Sum multiplied by the fame, the Product fhall be lefs than the Refolvend; which is fo far like what we do for the Square Root: But, again; the Remainder muft not exceed the Product of thefe Numbers, *viz.* the Sum of the Quote and the fecond Member of the Divifor multiplied into the Difference betwixt the Quote and its Square; *i e.* add together thefe two Products, and their Sum muft not exceed the Refolvend, and what remains here belongs to the next Refolvend.

You may alfo form your Divifor thus; Take the Figures already found, and to them prefix 1 (or take them with 0 prefix'd, and then add 1, which will fall in the Place of the 0); multiply this Sum by triple the Number to which the 1 was added : The Product is the Divifor. See this *Example* wrought after this Mannner.

$$
\begin{array}{l}
\qquad\qquad \overset{\cdot}{6}1\overset{\cdot}{4}125 \;(\overset{a.b}{8\,5} \\
\qquad\qquad 512 \;= a^3 \\[4pt]
\text{Divifor.} \\
3\,a^2 + 3\,a = 19440 \;) \quad 102125. \text{ 2d Refolvend.} \\
(= 3\,a \times \overline{a+1} = \\
\quad 3 \times 80 \times 81\,) \qquad 97225 = \overline{3\,a^2 + 3\,a + b} \times b \\
\qquad\qquad\qquad\qquad 4900 = \overline{3\,a + b} \times \overline{b^2 - b} \\[4pt]
\qquad\qquad\qquad 102125. \text{ Sum.} \\[4pt]
\qquad\qquad\qquad 000000
\end{array}
$$

The Letters and Operations fhew the Application of this Method; and what is to be demonftrated is only this, that theNumber compared to theRefolvend is equal to $3\,a^2 b + 3\,a\,b^2 + b^3$, which is the Number compared in the former Rule; and the Truth of this you'll find by performing the Operation of thefe two Products, and adding them thus, $\overline{3\,a^2 + 3a + b} \times b = 3\,a^2 b + 3\,a\,b + b^2$, then $\overline{3\,a + b} \times \overline{b^2 - b} = 3\,a\,b^2 + b^3 - 3\,ab - b^2$: which added to the former makes $3\,a^2 b + 3\,a\,b^2 + b^3$.

What I have further to obferve is, That this Method will in many Cafes be of Advantage, by helping us to difcover more eafily that fome Figures are too great for the Quote, without the Trouble of making out the total Number, which is here to be compared with the Refolvend : For if the firft Part of it (*viz.* the Product of the Quote by the Sum of the Divifor and Quote, or $\overline{3\,a^2 + 3a + b} \times b$) is equal to the Refolvend, or greater,

greater, that Figure is certainly too big to anfwer the Rule. But tho' that firft Part is lefs than the Refolvend, we cannot conclude that we have the true Figure, till we add the other Part alfo, and find that the Sum is not greater than the Refolvend.

Obferve alfo, That if the Product of the Quote *b*, and firft Member of the Divifor, *viz.* $3a^2$, is equal to the Refolvend or greater, then certainly that Quote is too big, and fo we might have the fame kind of Advantage by the common Method; yet the Product of the Quote into the Sum of the Quote and Divifor, being always a greater Number than the Product of the Quote and firft Member of the Divifor, the laft Method will difcover fome Figures to be too great, which would not appear fo without Trial by the other Method.

Of the Proof *of the Cube Root.*

Involve the Root found to the Cube, and compare it with the given Cube, or the Difference betwixt the given Number and the Remainder of the Extraction.

Or, By cafting out 9's thus : Caft the 9's out of the given Number, if there is no Remainder in the Extraction; or out of the Difference of that Number and the Remainder of the Extraction : Then caft the 9's out of the Root found, and fquare the Excefs, out of which caft the 9's, and multiply this Excefs by the preceding, and out of this Product caft the 9's; the Excefs or Defect of 9 muft be equal to that found in the given Number.

Example : The Cube Root of 2744 is 14; thus proved : The Excefs of 9's in 2744 is 8. in 14 it is 5; then $5 \times 5 = 25$, in which the Excefs of 9's is 7, which, multiplied by the preceding Excefs 5, the Product is 35, in which the Excefs of 9's is 8.

$14 \times 14 \times 14 = 2744$ | The Reafon of this Practice is alfo obvious from what is fhewn in Multiplication : For taking 14×14 as one Factor, and 14 as another, we firft caft out 9's out of 14×14, and then out of 14; and, multiplying thefe two Exceffes together, we compare the Excefs of 9's in the Product with that in 2744, which is $= \overline{14 \times 14} \times 14$.

Problem III.   *To Extract the* Root *of any Power above the* Cube.

General R U L E.

WHatever Root is propofed to be extracted, as in general the *n* Root, diftribute the given Number into Periods, taking as many Figures to each Period as the Number of Units in the Index *n*; then make a Table of the fimilar Powers (*i. e.* the *n* Powers) of all the Digits, as far at leaft till you find one which is equal to, or exceeds the firft Period on the Left of the given Number, taken by itfelf; the Root of that Power is the firft Figure on the Left of the Root fought, which call A; then fubtract A^n from the faid firft Period, and to the Remainder prefix the next Period for a Refolvend; and to find the next Figure of the Root, form a Divifor thus; take a Binomial $A + B$, and involve it to the *n* Power, as has been explained; your Divifor is the Sum of all the Products of the feveral Powers of A, except the higheft A^n, multiplied by the proper Coefficients of the Terms in which they ftand in the Power $\overline{A+B}^n$: Thus, for the 4th Root the Divifor is $4A^3 + 6A^2 + 4A$; for the 5th Root it is $5A^4 + 10A^3 + 10A^2 + 5A$, as you'll find from the *Table of Binomial Powers and Coefficients*. And *univerfally*, the Divifor will be $nA^{n-1} + a \times A^{n-2} + b \times A^{n-3} + \&c. + n \times A$, where I have fimply expreffed the Coefficients by fingle Letters, which you muft underftand as reprefenting the true Coefficients. Alfo remember, that the firft Figure of the Root found, which A reprefents, muft be mul-

tiplied

tiplied by 10, or o prefix'd to it, becaufe that is its true Value with refpect to the next Figure to be found; and in this Value you are to ufe it in forming the Divifor; then find the Quote B, (which can never exceed 9) limited fo that the feveral Members of the Divifor being multiplied by the feveral Powers of B, which you find multiplied into them in the feveral Terms of the Binomial Power $\overline{A+B}^n$; and B^n added to the Sum of all thefe Products, the total fhall not exceed the Refolvend. Thus in the 4th Power the Number to be compared with the Refolvend is this, $4 A^3 B + 6 A^2 B^2 + 4 A B^3 + B^3$; in the 5th Power it is $5 A^4 B + 10 A^3 B^2 + 10 A^2 B^3 + 5 A E^4 + E^5$. Univerfally, it is $n A^{n-1} \times B + a A^{n-2} \times B^2 + 6 A^{n-3} \times B^3 + \&c. + n A B^{n-1} + B^n$, which is the whole Binomial Power except the Term A^n.

In the next place take the two Figures found, and prefixing o, call this Number A, and form a new Divifor as before, of the feveral Powers of this new Number A, multiplied by their Coefficients in $\overline{A+B}^n$, and by this find the 3d Figure of the Root, which call again B, under the fame Limitation as before; and fo proceed to the End.

S C H O L I U M S.

1. If you begin your Guefs for the Quote (*i.e.* for any Figure of the Root after the firft) at the greateft number of times (not exceeding 9) that the Divifor is contained in the Refolvend, then the Limitation of the Rule for the Number to be compared with the Refolvend is fufficient to determine when you have the true Figure. But if you chufe at a Guefs, then you are to mind this Mark of a Figure too little, *viz.* That if you take all the Root found, taking in the Figure now put in the Root, and call it A; then take the Sum of the Products of the feveral Powers of it (except A^n) which belong to the new Divifor; the Remainder muft not exceed this, elfe the Figure laft found is too little.

2. If there is a Remainder after all the Periods are employed, the given Number is not a Power of that Order, and the Root found is only that of the greateft Power contained in it.

Roots.	5th Powers.
1 :	1
2 :	32
3 :	243
4 :	1024

I fhall illuftrate this Rule as far as is neceffary by an Example. Suppofe the 5th Root of this Number 74560898 is required. Having pointed it, the laft Period is 745; and raifing the 5th Powers of the Numbers from 1 to 4, whofe 5th Power is the firft which exceeds 745, I find 243 the greateft 5th Power contained in 745; and the 5th Root of this being 3, I put 3 as the laft Figure of the Root fought.

$$A^5 = \begin{array}{l} \dot{7}4\dot{5}60898 \\ 243 \end{array} \qquad \left(\begin{array}{l} A B \\ 3 \ 7 \end{array} \right.$$

$$4329150) \ 50260898. \ \text{2d Refolvend.}$$

$$28350000 = 5 A^4 \times B \qquad \left(\begin{array}{l} A = 30 \\ B = 7 \end{array} \right)$$
$$13230000 = 10 A^3 \times B^2$$
$$3087000 = 10 A^2 \times B^3$$
$$360150 = 5 A \times E^4$$
$$16807 = E^5$$

$$45043957. \ \text{Sum.}$$

$$5216941. \ \text{Remainder.}$$

Then raifing $\overline{A+B}^5$, it is $A^5 + 5 A^4 B + 10 A^3 B^2 + 10 A^2 B^3 + 5 A B^4 + B^5$; and taking $A = 30$, the Divifor is $5 A^4 + 10 A^3 + 10 A^2 + 5 A = 4329150$, which is found in the Refolvend, under the Limitation of the Rule 7 times; the Remainder being 5216941.

The

The Divisor being formed thus:

$$5 A^4 = 4050000$$
$$10 A^3 = 270000$$
$$10 A^2 = 9000$$
$$5 A = 150$$
$$\overline{4329150}$$

DEMONSTR. The Demonſtration of this general Rule depends upon the ſame kind of Principles as thoſe for the Square and Cube: And whoever underſtands theſe thoroughly will be able to extend them to this univerſal Rule with great Eaſe: For if we put n in the Place of 2 or 3 in the preceding *Lemma's*, they will become univerſal for all Caſes.

SCHOLIUM. What a tedious thing it is to form the Diviſors, and the Numbers to be compared with the Reſolvend in high Powers, and indeed in all above the Cube, it's eaſy to perceive. All that can be ſaid in favour of this general Rule is only this, That it is exceedingly preferable to our being left to a pure blind Gueſs, with no other Help than raiſing the Power of the Root gueſſed, and comparing it with the propoſed Number. Yet the great Labour of this Rule has excited the Mathematicians to the Invention of other Methods; the explaining of which comes not within the Limits I have preſcribed my ſelf in this Work, except that Method which is by the help of *Logarithms*, as you'll find afterwards explained. In the mean time *obſerve*, that as Square and Cube Roots are the things only uſeful in the common Affairs of Life, ſo the Rules for them are tolerably eaſy, eſpecially the Square. But there is alſo

Another General RULE *for* Compound *Roots* (i. e. *whoſe Index is the Product of two or more Numbers*).

Take any two or more Indexes whoſe Product is the given *Index*, and extract out of the given Number a Root anſwering to any of theſe leſſer Indexes; and then out of this Root extract a Root anſwering to another of theſe leſſer Indexes, and ſo on, till you go thro' them all: The laſt Root found is the Root ſought.

Example 1. To find the 4th Root of 625, I find the Square Root 25; then the Square Root of this, which is 5, is the Root ſought.

Example 2. To find the 6th Root of 4096: It is 4; which I find thus: $6 = 2 \times 3$, therefore I find the Square Root of 4096, which is 64, and then the 3d Root of 64 is 4.

Demonſtr. The *Reaſon* of this Rule is obvious, being only the Reverſe of what's done and demonſtrated for involving a Number to a compound Power; or you have the Reaſon of it in *Theor*. IX. §. 1. where it's ſhewn that $A^{\frac{1}{nm}} = A^{\frac{1}{m}\cdot\frac{1}{n}}$.

Obſerve, It's beſt to begin with the Root of the loweſt Index.

Alſo, If the given Number is not a Power of the Order you firſt try, neither can it be a Power of the Order propoſed; and to find the Root of the greateſt like Power contained in it, other Methods do better.

Of the Proof *of all Roots of Integers univerſally.*

It is done either by the oppoſite Involution, or by caſting out the 9's, thus:
Caſt the 9's out of the given Number, or the Difference of it and the Remainder of the Extraction, and mark the Exceſs: Then caſt the 9's out of the Root (and take the Exceſs, or the Root itſelf if leſs than 9); multiply it by itſelf, and caſt out the 9's from the Product; then multiply the Exceſs by the Exceſs in the Root, and caſt the 9's out of the Product; this laſt Exceſs multiply by the Exceſs in the Root, and caſt the 9's out of the Product, and go on ſo till the Exceſs of 9's in the Root is employ'd as a Multiplier, as

oft

oft as the Index of the Power expresses: The last Excess must be equal to that in the given Number.

§. II. P a r t II.

Probl. 4. *Of the* Extraction *of the Roots of* Fractions.

A Fractional Power is to be considered in two different Views: 1. As being an immediate Power, *i. e.* the immediate Effect of the continual Multiplication of some Fraction into itself, as $\frac{4}{9} = \frac{2}{3} \times \frac{2}{3}$; and $\frac{8}{27} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$. Or, 2. As being only equivalent to some immediate Power, but not itself such a one, as $\frac{8}{18} = \frac{4}{9}$.

Now if a Fraction is immediately a Power, it's manifest from the Definitions, that if we extract the Root proposed from the Numerator and Denominator separately, these are the Numerator and Denominator of the fractional Root sought. *Example*: The Square Root of $\frac{64}{81}$ is $\frac{8}{9}$; for $8 = \overline{64}^{\frac{1}{2}}$, and $9 = \overline{81}^{\frac{1}{2}}$. But if the given Fraction is only equivalent to some immediate Power, the Root (*viz.* of that Power; which is also in another Sense, the Root of the given Fraction) cannot be discovered by this Method; for the Numerator and Denominator have not both in this Case, and perhaps neither of them has a perfect Root; and so we cannot determine by this Method, whether the Root sought is rational or surd: yet by other Methods we can discover this, and find the Root where there is one. For which take this

General R u l e.

Reduce the given Fraction to its lowest Terms, and then extract the proposed Root from Numerator and Denominator separately; and these Roots are the Numerator and Denominator of the Fractional Root sought; which is also in its lowest Terms. But if both Numerator and Denominator have not such a perfect Root, the given Number is not a Power of the Order proposed, either immediately or equivalently.

Example 1. To find the Square Root of $\frac{27}{75}$, I find its least Terms $\frac{9}{25}$, whose immediate Root is $\frac{3}{5}$.

Example 2. To find the Square Root of $\frac{24}{69}$, I find its least Terms $\frac{8}{23}$. ' But neither 8 nor 23 are Squares, and therefore $\frac{24}{69}$ is not a Square in any Sense.

Demon. 1. If the lowest (or any) Terms of the given Fraction are Powers of the Order proposed, it's plain that their Roots make a Fraction, which is the Root of the given Fraction; by the Definition. And,

2. If the least Terms of a Fraction are not Powers of the given Order, no Terms of it are so; or the given Fraction is not a Power in any Sense. For let $\frac{A}{B}$ be a Fraction in its least Terms, and suppose $\frac{M^n}{N^n} = \frac{A}{B}$ (*i. e.* some other equivalent Terms of the Fraction to be an immediate Power.) Then because $\frac{A}{B}$ is in least Terms, $\frac{M^n}{N^n}$ is not so, because it consists of different Terms by Supposition. Consequently $\frac{M}{N}$, its *n* Root, is not

in its leaft Terms, (by *Lem.* preceding the *Theor.* Chap. 1.) Take $\frac{r}{s}$ in its leaft Terms, and

$=\frac{M}{N}$; then is $\frac{r^n}{s^n}$ in its leaft Terms, (by the fame *Lem.*) And fince $\frac{r}{s}=\frac{M}{N}$, therefore $\frac{r^n}{s^n}$

$=\frac{M^n}{S^n}=\frac{A}{B}$; wherefore $\frac{r^n}{s^n}$ and $\frac{A}{B}$ are both in the leaft Terms, which is abfurd; or $\frac{r^n}{s^n}$, $\frac{A}{B}$

are the fame Terms, which is alfo contrary to Suppofition.

S C H O L I U M S.

1. A Fraction made of the greateft Integral Root of the Numerator and Denominator may in one Senfe be called the Root of the greateft Fractional Power contained in the given Fraction; which Root will, in fome Cafes, be a deficient, and in fome an exceffive Root, *i.e.* whofe Power wants of, or exceeds the given Fraction. *Example:* $\frac{5}{13}$, the greateft Square Fraction contained in it, in this Senfe, is $\frac{4}{9}$, whofe Root is $\frac{2}{3}$; which is an exceffive Root to $\frac{5}{13}$, becaufe $\frac{4}{9}$ is a greater Fraction than $\frac{5}{13}$. But in $\frac{8}{15}$ the greateft Square is $\frac{1}{9}$, which is lefs than $\frac{3}{15}$; therefore its Root $\frac{1}{3}$ is a deficient Root to $\frac{3}{15}$.

Again: In another Senfe, *i. e.* if we ask what is the greateft Fraction which is an immediate Power, and is lefs than a given Fraction which is not a Power in any Senfe, then there is no fuch thing as a greateft; the Reafon of which you'll find afterwards. (See *Corol. Prob.* 5.)

2. The preceding General Rule requires two Extractions, *viz.* both from the Numerator and Denominator; but I fhall give you other particular Rules, whereby the Root is found by one Extraction; and fuch as are accommodated to the Methods of Approximation, afterwards explain'd.

Particular R U L E s *for the Roots of Fractions.*

1. *For the Square Root.*

Multiply the Numerator and Denominator together, and extract the Square Root of the Product; which is always a compleat Square, if the given Fraction is fo in any Senfe. Make this Root the Numerator to the given Denominator, and this Fraction is the Root fought; or fet the given Numerator fractionally over the Root found; and this alfo is the Root fought, tho' neither of them is in the leaft Terms. But if the Product is not a compleat Square, neither is the given Fraction: And having found the Root of the greateft Integral Square contained in it, ufe that as directed; and you fhall have a Root wanting of a juft Root to the given Fraction, if the Root extracted is made Numerator; but exceeding, if it's made Denominator.

Example 1. To find the Square Root of $\frac{4}{9}$, I take $4\times 9=36$, whofe Root is 6; and fo the Root fought is $\frac{6}{9}=\frac{2}{3}$, or $\frac{4}{6}=\frac{2}{3}$; for $\frac{2}{3}\times\frac{2}{3}=\frac{4}{6}$.

Example 2. To find the Square Root of $\frac{12}{147}$, I take $12\times 147=1764$, whofe Root is 42; and fo $\frac{42}{147}=\frac{12}{42}=\frac{2}{7}$ is the Root fought; for $\frac{2}{7}\times\frac{2}{7}=\frac{4}{49}=\frac{12}{147}$.

B b

Example

Example 3. For the Square Root of $\frac{7}{12}$, I take $7 \times 12 = 84$, which is not a Square; therefore $\frac{7}{12}$ is not so: But the greateſt Integral Root in 84 is 9, therefore $\frac{9}{12} = \frac{3}{4}$ is a Root wanting of a true Root to $\frac{7}{12}$; for $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$, which is leſs than $\frac{7}{12}$, becauſe 9×12 is leſs than 7×16. And $\frac{7}{9}$ is an exceſſive Root; for $\frac{7}{9} \times \frac{7}{9} = \frac{49}{81}$ greater than $\frac{7}{12}$, becauſe 49×12 is greater than 7×81.

SCHOLIUM. Tho' the Fraction given be in its leaſt Terms, yet the Root found by this Method will not be in its leaſt Terms in every Caſe where the given Fraction is not a perfect Power, as the preceding *Exam.* 3. ſhews. And if it is a perfect Power, the Root found will never be in its leaſt Terms, as is manifeſt; becauſe the leaſt Terms are the Root of the Numerator and Denominator of the leaſt Terms of the given Fraction.

DEMON. 1. The given Fraction being $\frac{A}{B}$, multiply both Terms by B, or by A; and then $\frac{A}{B} = \frac{AB}{BB} = \frac{AA}{AB}$. Suppoſe AB is a compleat Square, whoſe Root is a, ſo that AB $= aa$; then are all theſe Expreſſions equal, *viz.* $\frac{aa}{BB} = \frac{AB}{BB} = \frac{A}{B} = \frac{AA}{AB} = \frac{AA}{aa}$; conſequently $\frac{a}{B}$ the Square Root of $\frac{aa}{BB}$, and $\frac{A}{a}$ the Square Root of $\frac{AA}{aa}$ are each a true Root to $\frac{A}{B}$.

2. If AB is not a perfect Square, then neither is $\frac{A}{B}$ in any Senſe. For ſuppoſe $\frac{M^2}{N^2} = \frac{A}{B} = \frac{AB}{BB}$, then, by equal Multiplication, according to the Nature of Fractions, it is $\frac{M^2 \times B^2}{N^2} = AB$. But $M^2 \times B^2 = \overline{MB}^2$, therefore $\frac{M^2 \times B^2}{N^2} = \frac{\overline{MB}^2}{N^2}$; which is plainly an immediate Square, whoſe Root is $\frac{MB}{N}$, and is therefore a true Root to AB, which is abſurd; for AB is ſuppoſed not to have a true Square Root in Integers, and conſequently has no ſuch Root true, (*Theor.* 19. *Chap.* 1.) therefore $\frac{MB}{N}$ is not the Square Root of AB; Nor is $\frac{M^2 \times B^2}{N^2}$ equal to AB; nor $\frac{M^2}{N^2}$ equal to $\frac{AB}{BB} \left(= \frac{A}{B} \right)$ as was ſuppoſed, *i. e.* no immediate Square $\frac{M^2}{N^2}$ can be equal to $\frac{A}{B}$, or $\frac{A}{B}$ is not a Square in any Senſe.

3. Suppoſe $\bar{a}$ the greateſt Integral Root of AB, ſo that aa is leſs than AB; then is $\frac{aa}{BB} \left(\frac{A}{B} \right)$ *i. e.* $\frac{a}{B}$ is a deficient Root to $\frac{A}{B}$. Alſo ſince aa is leſs than AB, therefore $\frac{AA}{aa}$ is greater than $\frac{AA}{AB} \left(= \frac{AB}{BB} = \frac{A}{B} \right)$ *i. e.* $\frac{A}{a}$ is an exceſſive Root to $\frac{A}{B}$.

2. For the Cube Root.

Multiply the Numerator by the Square of the Denominator; the Product will be a compleat Cube, if the given Fraction is ſo in any Senſe: the Cube Root of this Product ſet fractionally over the Denominator of the given Fraction, is the fractional Root ſought.

fought. Or *thus*: Multiply the Denominator by the Square of the Numerator, and over the Cube Root of the Product set the given Numerator fractionally; and this also is the Root fought. And *observe*, that if the given Fraction is a proper one, the last is the best Method; but if it is improper, take the first Method. But if the Product mention'd is not a compleat Cube, neither is the given Fraction. And if we take the greatest Integral Root of that Product, and use it as directed, we have a deficient or excessive Root, according as we chuse the first or second Method.

Example 1. To find the Cube Root of $\frac{8}{27}$, take $27 \times 27 = 729$, then $729 \times 8 = 5832$, whose Cube Root is 18; then is $\frac{18}{27} = \frac{2}{3}$ the Root fought. Or thus; $8 \times 8 = 64$, then $64 \times 27 = 1728$, whose Cube Root is 12; therefore $\frac{8}{12} = \frac{2}{3}$ is also the Root fought.

Example 2. For $\frac{15}{54}$, take $54 \times 54 = 2916$, then $2916 \times 15 = 43740$, which has not a Cube Root, and so $\frac{15}{54}$ is not a Cube; but the greatest integral Cube Root of 43740 being 35, therefore $\frac{35}{54}$ is a deficient Root to $\frac{15}{54}$; or if we take $15 \times 15 = 225$, then $225 \times 54 = 12150$, whose nearest Cube Root is 22, and $\frac{15}{22}$ is an Excessive Root.

DEMON. 1. For the first Method, multiply each Term of the Fraction $\frac{A}{B}$ by B^2, and it is $\frac{A B^2}{B^3} = \frac{A}{B}$; and if $A B^2$ is a Cube, let its Root be m, then $\frac{m}{B}$ is the Cube Root of $\frac{A B^2}{B} = \frac{A}{B}$. Again; if $A B^2$ is not a Cube, neither is $\frac{A}{B}$; for if we suppose $\frac{m^3}{n^3} = \frac{A B^2}{B^3}$, then is $A B^2 = \frac{m^3 \times B^3}{n^3} = \frac{\overline{m B}^3}{n^3}$, whose Cube Root is $\frac{m B}{n}$; *i. e.* $A B^2$ is a Cube, which is contrary to Suppofition, if $\frac{m B}{n}$ is Integer; and if it's not Integer, it cannot be the Cube Root of $A B^2$, which has no Cube Root in Integers (*Theor.* XIX. *Ch.* 1.). But if we suppose m the greatest integral Cube Root contained in $A B^2$, so that m^3 is less than $A B^2$, it's plain that $\frac{m^3}{B^3}$ is less than $\frac{A B^2}{B^3} = \frac{A}{B}$, or $\frac{m}{B}$ is a deficient Root.

2. For the fecond Method, multiply each Term of $\frac{A}{B}$ by A^2, and it is $\frac{A^3}{A^2 B} = \frac{A}{B}$, so that $A^2 B$ being a true Cube, whose Root is n, then $\frac{A}{n}$ is the Root of $\frac{A^3}{A^2 B} = \frac{A}{B}$; but if $A^2 B$ is not a Cube, neither is $\frac{A}{B}$; for suppose $\frac{m^3}{n^3} = \frac{A^3}{A^2 B}$, then is $\frac{m^3}{n^3} \times A^2 B = A^3$, and $A^2 B = A^3 \div \frac{m^3}{n^3} = \frac{A^3 \times n^3}{m^3} = \frac{\overline{A n}^3}{m^3}$, whose Cube Root is $\frac{A n}{m}$; *i. e.* $A^2 B$ is a Cube, contrary to Suppofition: But if we take m the greatest integral Cube Root contained in $A^2 B$, so that m^3 is less than $A^2 B$, then is $\frac{A^3}{m^3}$ greater than $\frac{A^3}{A^2 B}$, and consequently $\frac{A}{m}$ is an excessive Root to $\frac{A}{B}$.

SCHOLIUM. Both thefe, and the Extraction of all higher Roots, may be comprehended in one general *Rule*, thus:

Gene-

General RULE *for all the Roots of* Fractions, *after the Manner of the preceding* Particular *Rules.*

Raise the Denominator of the given Fraction to a Power, whose Index is 1 less than that of the Root to be extracted, and multiply this Power by the given Numerator: Extract the proposed Root of this Product (which is a compleat Power, if the given Fraction is so), and set it fractionally over the given Denominator, and this makes the Root sought. But if that Product has not such a Root, neither has the given Fraction; and taking the greatest integral Root contained in that Product, it makes, with the given Denominator, a deficient Root to the given Fraction.

Or also thus: Raise the Numerator to the Power directed, and by that multiply the Denominator; extract the proposed Root of the Product if it has one, or take the greatest integral Root contained in it; over this Root set the given Numerator, and it makes an excessive Root to the given Fraction.

Example. To find the 4th Root of $\frac{A}{B}$, I multiply B^3 by A, and set the 4th Root of the Product AB^3, or the greatest integral 4th Power contained in it, over B, and it makes the Root sought, or a deficient one; or also over the 4th Root of A^3B set A, it makes the Root sought, or an excessive one.

DEMON. 1. For the n Root of $\frac{A}{B}$: If AB^{n-1} is a true Power of the Order n, let its Root be m, then it's plain that $\frac{m}{B}$ is the n Root of $\frac{AB^{n-1}}{B^n} = \frac{AB^{n-1}}{B \times B^{n-1}} = \frac{A}{B}$. And if m is only the Root of the greatest integral Power contained in AB^{n-1}, it self not being one, then is $\frac{m}{B}$ plainly a deficient Root to $\frac{A}{B}$, which in this Case has no true Root; for if we suppose $\frac{m^n}{n^n} = \frac{AB^{n-1}}{B^n}$ $(= \frac{A}{B})$, then is $A B^{n-1} = \frac{m \times B^n}{n^n} = \frac{\overline{mB}^n}{n^n}$, whose n Root is $\frac{mB}{n}$; *i.e.* AB^{n-1} is a Power of the Order n, contrary to Supposition.

2. For the 2d Method; $\frac{A^n}{BA^{n-1}} = \frac{A}{B}$, and if the n Root of BA^{n-1} is n, then is $\frac{A}{n}$ the Root sought: But if n is only the Root of the greatest integral Power contain'd in BA^{n-1}, it self not being one, then is $\frac{A}{n}$ an excessive Root to $\frac{A}{B}$, which in this Case has no true Root; for if $\frac{m^n}{n^n} = \frac{A^n}{BA^{n-1}}$, then is $BA^{n-1} = \frac{A^n \times n^n}{m^n}$, whose n Root is $\frac{A^n}{n}$; *i.e.* BA^{n-1} is a Power of the Order n, contrary to Supposition.

SCHOLIUM. If the Denominator of the Root is a Compound Number; *i.e.* the Product of two or more Integers, the Extraction may be made by several more simple Extractions, in the manner already explained, which needs not to be further insisted on.

§. II. Part III.

Problem 5. *Of the* Approximation *of* Roots.

Definition.

WE have already obſerved, that tho' a Number has no determinate Root, yet it has what we may call an indeterminate one (ordinarily called a *Surd* Root); *i. e.* there is a certain Series of Numbers decreaſing, which can be carried on by a certain Law or Order *in infinitum,* whoſe Sum taken from the beginning is a Root whoſe Power approaches nearer and nearer to the given Number, as the Series goes on; and tho' it is never equal to it preciſely, it may be brought within any aſſignable Difference: The *Invention,* or carrying on of this Series is what we here call the *Approximation* of the Root; and if we take the Series of the Sums invented, it may be called the Series of *Approaching* Roots. Obſerve alſo, that they may be found approaching yet either ſtill leſs or ſtill greater than true Roots.

I. *For Roots of* Integers.

Rule. Whatever Root is propoſed, after the Root of the greateſt integral Power contained in the given Number is found, by the preceding Rules; To the Remainder prefix a Period of o's according to the Index; thus oo for a Square Root, ooo for a Cube, oooo for a 4th Power, and ſo on: Then form a Diviſor, and find a new Figure in the Root the ſame way as in the preceding Steps of the Work: To every ſucceeding Remainder prefix a Period of o's, and find a new Figure of the Root, and this Work will go on for ever, becauſe there will always be a Remainder. The Figures thus found are all Decimal Places in the Root, the decimal Point being placed immediately after the integral Part, and before theſe new Figures. And thus we have a Mixt Number for the Root; which is ſtill nearer and nearer to the true Root of the given Number, the further the Operation is carried on, but is ſtill deficient, becauſe there is ſtill a Remainder. Again; *Obſerve,* that if to the laſt Figure found in the Root you add 1, the Sum will make an exceſſive Root; and thus you may have a Series of Roots nearer and nearer, but ſtill exceſſive. The following Example of a Square Root will ſufficiently illuſtrate this Practice.

Operation.

```
387 (19.672, &c.
  1
  ---
287
261
  ---
2600
2316
  ---
 28400
 27489
  ---
  91100
  78684
  ---
  12416
   &c.
```

To find the Square Root of 387. The Root of the greateſt integral Square contained in it is 19. Then by one Period of o's the Root becomes 19.6; by a 2d it is 19. 67; by a 3d it is 19. 672; and may be carried further at pleaſure; and each of theſe Roots are deficient; *i. e.* their Figures are leſs than 387; but the Difference is ſtill leſs and leſs: and what I called the Series of Numbers decreaſing, whoſe Sums make the Series of approaching Roots, tho' ſtill defective, are theſe, 19, .6, .07, .002, *&c.* and the Series of their Sums, which make the approaching Root, is 19, 19.6, 19.67, 19. 672, *&c.* And laſtly, by adding 1 to each of theſe, we have a Series of approaching Roots, but ſtill exceſſive, tho' the Differences grow ſtill leſs. Thus; 20, 19.7, 19.68, 19.673, *&c.*

Demon. 1. If any compleat integral Power of any Order is multiplied into a Number which is not a Power of that Order, the Product is not a Power of that Order; or has not a perfect Root of that Order. Thus; If A is not a Power of the Order *n,* neither is A × B^n, as has been demonſtrated in *Theor.* II. *Coroll.* 4. *Chap.* I.

2. If

2. If the Root of the greateſt integral Power contained in AB^n is divided by B, which is the n Root of the Multiplier B^n, the Quote is leſs than a true Root to the given Number. For ſuppoſe r to be the n Root of the greateſt integral Power of the Order n contained in AB^n, and it's plain that r^n is leſs than AB^n, therefore take their like aliquot Parts, and $\frac{r^n}{B^n}$ is leſs than $\frac{AB^n}{B^n}$, or A; *i.e.* $\frac{r}{B}$, the n Root of $\frac{r^n}{B^n}$, is leſs than a true Root to A. Again; If to r, the greateſt integral Root of AB^n, we add 1, and call the Sum s, then s^n is greater than AB^n; and conſequently $\frac{s^n}{B^n}$ greater than $\frac{AB^n}{B^n}$, or A; *i.e.* $\frac{s}{B}$ is an exceſſive Root to A.

From theſe two Articles we ſhall eaſily-demonſtrate the Rule of *Approximation,* thus:

3. The greateſt integral Root, or Root of the greateſt integral Power contained in the given Number being found, what remains to be proved is this only, That the Extraction will go on in this manner without end; *i.e.* that there will always be a Remainder, and ſo a new decimal Fraction will at every Step be added to the preceding Root, making the whole greater and greater; yet ſo that the Mixt Root will ſtill be deficient, or its Power ſtill leſs than the given Number, tho' ſtill nearer *in infinitum.* To ſhew this Truth, *conſider,* that by prefixing Periods of o's to any Number, we do really multiply it by a Number conſiſting of 1 with as many o's as are thus prefix'd; but it's the ſame thing to multiply the given Number (whoſe Root we ſeek) by prefixing o's, and then bringing them down to the Remainders, or prefixing them only to the Remainders; for either way we find the Root of the Product (or the greateſt integral Root contained in it). Thus, for a Square Root one Period oo multiplies the given Number by 100, two Periods multiplies by 10000, *&c.* For a Cube Root one Period ooo multiplies by 1000, and two Periods multiplies by 1000000, *&c.* and ſo of other Powers. But theſe Multipliers are evidently true and compleat Powers of their ſeveral Orders, whoſe Roots are 1, with as many o's as we have uſed Periods of o's; therefore, by the firſt Article, however far the Extraction is carried by Periods of o's thus prefix'd to the Remainders, *i.e.* however great the Power is by which we have thus multiplied the given Number, there will always be a Remainder, becauſe the given Number not being a true Power, tho' the Multiplier is, yet the Product is not. Again; By putting all the Figures found by means of theſe Periods of o's, in decimal Places, we do evidently divide the Root of the Product, (*i.e.* the Root of the greateſt integral Power contain'd in it) by the Root of the Number multiplied into the given Number: Becauſe for every Period annex'd we have one Place in Decimals; which is plainly dividing the Root found, conſidered all as a whole Number, by 10, or 100, *&c.* according to the Number of Periods of o's employ'd: Therefore, by the 2d Article, this Mixt Root will always be leſs than a juſt Root to the given Number, tho' ſtill approaching nearer, which demonſtrates the Rule as to the Series of deficient Roots: and as to the exceſſive Roots, it's evident that adding an Unit to the laſt Place of the Root already found, is adding 1 to the Root of the greateſt integral Power contained in the given Number, or to its Product by the Power which multiplies it: Therefore, by the 2d Article, the Root becomes exceſſive. Or it's found by this Conſideration, That 1 in any Place of a Number either integral or decimal, is of more value than all the reſt of the Number ſtanding on the Right of that Place, however many Figures there be.

SCHOLIUMS.

1. The Proof of this Operation is made the ſame way as has been already explain'd, *viz.* either by raiſing the Root found to its Power, and adding the Remainder; or by caſting out of the 9's.

As to the former Method, *Obſerve,* That we need to take no notice of the Root's being a Mixt Number, but take it all as a whole Number, and the Remainder ſo alſo; and then

then the Sum of the Power and Remainder muft have as many Periods of o's on the Right as were ufed in the Operation; becaufe when the Root and Remainder are taken for Integers, fo many Periods of o's belong to the fuppofed Power or Number, whereof that Root is the greateft integral Root: But if we take the Root as it really is, a Mixt Number, then the Remainder is a decimal Fraction, whofe Denominator is 1, with as many o's as were added to all the Remainders in the Operation, and in this value it is to be added to the Power of the Mixt Root: Thus in the preceding Example, the Root found is 19.672, whofe Square is 386.987584, and the Remainder is, in its true value, .012416; for two Periods, or 6 o's, were employed in the Operation; and the Sum of 386.987584, +.012416 is = 387.000000 (= 387), which is the fame as if the Quote and Remainder had been taken for Integers, and the given Number had been 387000000.

As to the Method by cafting out 9's; When we fubtract the Remainder from the given Number, we may take it either in its real Value, or as a whole Number, and then we muft take the given Number, with as many o's after it as were ufed in the Operation. For it is the fame to the prefent Purpofe, to take .012416 from 387, or 12416 from 387000000, the Remainder in both Cafes being the fame Figures, *viz.* 386.987584, or 386987584.

2. If we point the true Value of the Remainder at every Step of the Approximation, this will fhew gradually how much the propofed Power of the Root found wants of the given Number; and as the Root, fo confequently its Power continually increafes; therefore thefe Remainders will continually diminifh; fo that by obferving this, we can carry on the Work till that Difference or Remainder be as little as we pleafe, or lefs than any affigned Difference.

But if inftead of this, it fhould be required to extract the Root fo near to a true and perfect Root to the given Number that it fhall want lefs than an affigned Difference, *i. e.* fo that this Difference added to the Root found, the Power of the Sum fhall exceed the given Number, it's done thus; Suppofe any Fraction $\frac{a}{r}$ to be the given Difference, within which the Root is to be brought; then extract the Root to a Number of decimal Places equal to the Number of Figures in r, and you have done; for $\frac{1}{r}$ is lefs than $\frac{a}{r}$, if a is greater than 1; and a decimal Denominator having as many o's as r has Figures, is a greater Number than r; and fo a Fraction whofe Numerator is 1, and its Denominator that decimal one, is lefs than $\frac{a}{r}$, becaufe the Denominator is greater, and the Numerator not. Laftly, fince, as has been fhewn, 1 added to the laft Figure of the Root would make it exceed a true Root; therefore, in whatever Place of Decimals the laft Figure of the Root ftands, the whole does not want of a true Root to the given Number, an Unit of the Value of that Place, and confequently, if the Denominator of that laft Place is a Number greater than r, the Root is within $\frac{1}{r}$ of a true Root, becaufe it's within a leffer Fraction, and much more is it within $\frac{a}{r}$, which is greater than either of the former. Example: Let r be a Number of three Figures; if the Root have three decimal Places whereby the Denominator is 1000, the Root is within $\frac{1}{1000}$, which is lefs than any proper Fraction whofe Denominator is a Number of three Figures; fo in the preceding Example 19.672 is within $\frac{1}{1000}$ of a perfect Root to 387.

II. *For*

II. *For the Roots of* Fractions.

The Approximation of the Roots of Fractions is performed thus: Let that Root which the particular Rules for Square and Cube, or the general Rule following thefe, prefcribes to be extracted, be carried on to what Length of decimal Places you pleafe, and then divide it by the Denominator of the given Fraction, if you chufe the firft Method of thefe Rules; and thus you have a Root ftill lefs, but approaching to a perfect one: But if you chufe the 2d Method of the Rule, divide the given Numerator by that Root, and you have a Root approaching, but ftill exceffive; and the further the Approximation of that Root is carried, the truer will each of thefe fractional Roots be.

Example. For the Square Root of $\frac{13}{24}$, I take $13 \times 24 = 312$, whofe Root to 2 Places of decimals is 17.66, which divided by 24 Quotes $\frac{1766}{2400} = \frac{883}{1200}$, lefs than a true Root; or it is $13 \div 17.66 = \frac{1300}{1766} = \frac{650}{883}$, greater than a true Root.

D E M O N. The Reafon is manifeft from the preceding Rules; for the Square Root of $\frac{A}{B}$ is $\overline{AB}^{\frac{1}{2}} \div B$, or $A \div \overline{AB}^{\frac{1}{2}}$. Univerfally, the *n* Root of $\frac{A}{B}$ is $\overline{AB^{n-1}}^{\frac{1}{n}} \div B$, or $A \div \overline{B A^{n-1}}^{\frac{1}{n}}$ if thefe Roots are perfect; and if they are not, yet by approximating them we make the fractional Root alfo truer, tho' never perfect.

C O R O L L. Tho' a given Integer is not a perfect Power of any Order, yet there is a greateft Power of that Order, which is a leffer Number than the given one; and alfo there is a leaft Power of the fame Order, which is a greater Number than the given: But in Fractions there is no fuch greateft and leaft Power; becaufe we can find new Roots increafing for ever, or decreafing, yet fo as the Powers are ftill lefs or greater than the given Number.

S C H O L I U M. There remains one curious Problem relating to the Extraction of Roots, which goes a little deeper into the Algebraick Art than at firft I defigned: but without it, I found I muft omit feveral other curious things: and fince among feveral ways of folving this Problem, there is one that arifes very eafily and naturally from the Confideration of Square Numbers, efpecially the Square of a Binomial Root (already fufficiently explained) therefore I was determined to give it a place here.

P R O B L E M VI.

Having the Sum or Difference of any Square Number, and a certain Multiple of the Root; alfo having the Multiplier of the Root; to find the Root. Thus: Suppofe $R = a^2 + ac$, or $R = a^2 - ac$, or $R = ac - a^2$. Then if the Numbers expreffed by R and *c* are given, we can find the Number expreffed by *a* by the following Rules.

C A S E I.

When the Sum and Multiplier are given to find the Root, *i. e.* if $R = a^2 + ac$; and R, *c* are given to find *a*.

Rule. To the Sum add the Square of half the Multiplier, (or a 4th of the Square of the Multiplier.) Extract the Square Root of this Sum; and from it fubtract half of the Multiplier, the Remainder is the Root fought. Which Rule is expreffed in Characters thus:

$$a = \overline{R + \frac{cc}{4}}^{\frac{1}{2}} - \frac{c}{2}.$$

Example. $R = 21$, $c = 4$; then is $a = 3$; for $4 \times 4 = 16$, whose 4th is 4; then $21 + 4 = 25$, whose Square Root is 5, from which take 2 ($=$ the half of the Multiplier 4) the Remainder is 3 the Root. *Proof:* $3 \times 3 = 9$, $3 \times 4 = 12$, and $12 + 9 = 21$.

DEMON. Since by Suppofition, $R = a^2 + ac$; add $\frac{cc}{4}$ to each Side, then $R + \frac{cc}{4} = a^2 + ac + \frac{cc}{4}$; which laft Expreffion is a compleat Square, whofe Root is $a + \frac{c}{2}$: therefore $a + \frac{c}{2} = \overline{R + \frac{cc}{4}}|^{\frac{1}{2}}$ (*Ax.* 1.) and fubtracting $\frac{c}{2}$ from both, it is, $a = \overline{R + \frac{cc}{4}}|^{\frac{1}{2}} - \frac{c}{2}$. Which is the Rule.

C A S E II.

If the Difference and Multiplier are given, to find the Root;
Here there are two Rules, according as the Square or Multiple is fuppofed to be greateft.

1. Suppofe the Square greater than the Multiple, *i. e.* $R = a^2 - ac$.

RULE. To the Difference, add the 4th of the Square of the Multiplier; and to the Square Root of the Sum, add half the Multiplier: this Sum is the Root fought. Thus:

$$a = \overline{R + \frac{cc}{4}}|^{\frac{1}{2}} + \frac{c}{2}.$$

Example. $R = 28$, $c = 3$; then is $a = 7$: for $\frac{cc}{4} = \frac{9}{4} = 2\frac{1}{4}$, and $R + \frac{cc}{4} = 28 + 2\frac{1}{4} = 30\frac{1}{4} = \frac{121}{4}$, whofe Square Root is $\frac{11}{2}$ or $5\frac{1}{2}$; to which add $\frac{3}{2}$ or $1\frac{1}{2}$, the Sum is 7.

Proof. $7 \times 7 = 49$, and $3 \times 7 = 21$; then $49 - 21 = 28$.

DEMON. Since $R = a^2 - ac$, add $\frac{cc}{4}$ to both Sides; then is $R + \frac{cc}{4} = a^2 - ac + \frac{cc}{4}$. Which laft Expreffion is the Square of $a - \frac{c}{2}$. Wherefore $a - \frac{c}{2} = \overline{R + \frac{cc}{4}}|^{\frac{1}{2}}$; and adding $\frac{c}{2}$ to both Sides, it is $a = \overline{R + \frac{cc}{4}}|^{\frac{1}{2}} + \frac{c}{2}$. *Obferve,* Tho' $a^2 - ac + \frac{cc}{4}$ is the Square either of $a - \frac{c}{2}$, or $\frac{c}{2} - a$, yet we cannot here ufe $\frac{c}{2} - a$; for if a is lefs than $\frac{c}{2}$, a^2 is lefs than ac, contrary to Suppofition.

2. Suppofe the Multiplier greater than the Square, *i. e.* $R = ac - a^2$.

RULE. From the 4th of the Square of the Multiplier fubtract the given Difference, (which cannot exceed the Multiplier, if the Problem is poffible); then extract the Square Root of the Remainder; and either add it to, or fubtract it from half the Multiplier, (which is greater than the other, if the Problem is poffible); the Sum or Difference will either of them folve the Problem. Thus: $a = \frac{c}{2} + \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$, or alfo $a = \frac{c}{2} - \overline{\frac{cc}{4} - R}|^{\frac{1}{2}}$.

Example. $R = 6$, $c = 5$; then is $a = 3 = \frac{5}{2} + \overline{\frac{25}{4} - 6}|^{\frac{1}{2}}$, or $\frac{5}{2} + \overline{\frac{1}{4}}|^{\frac{1}{2}} = \frac{5}{2} + \frac{1}{2} = \frac{6}{2}$.

Proof. $ac = 15$, and $ac - a^2 = 15 - 9 = 6 = R$. Alfo, $a = 2 = \frac{5}{2} - \frac{1}{2} = \frac{4}{2}$. *Proof.* $ac - a^2 = 10 - 4 = 6 = R$.   C c   DEMON.

DEMON. Since $R = ac - a^2$. Subtract each of thefe from $\frac{cc}{4}$, then is $\frac{cc}{4} - R = \frac{cc}{4} - ac + a^2$; which laft is the Square either of $a - \frac{c}{2}$, or $\frac{c}{2} - a$. Wherefore $a - \frac{c}{2}$, or $\frac{c}{2} - a$ (according as a is greater or leffer than $\frac{c}{2}$) is $= \overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$. Hence, in the 1ft *Cafe*, or taking $a - \frac{c}{2}$; by adding $\frac{c}{2}$ to both Sides, it is $a = \frac{c}{2} + \overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$; and taking $\frac{c}{2} - a$, add $a - \overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$ to both Sides, it is $a = \frac{c}{2} - \overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$.

There remains yet to be demonftrated, That $\frac{cc}{4}$ can never be lefs than R, if the Problem is poffible; and that $\frac{c}{2}$ is greater than $\overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$. Now it is plain, that the Solution is impoffible, according to this Rule, if R is greater than $\frac{cc}{4}$; and that if R does not exceed $\frac{cc}{4}$, one of the Solutions is good. But to fhew that the Problem will always neceffarily have the two Solutions explained, it muft be fhewn that R cannot exceed $\frac{cc}{4}$, when it is $= ac - a^2$, and then the other Part will eafily follow.

To demonftrate this, we muft firft *obferve*, That a may be either greater or leffer than $\frac{c}{2}$ confiftently enough with $R = ac - aa$. For this requires no more than that ac be greater than aa, which requires again that a be lefs than c; confequently, whether a be greater or lefs than $\frac{c}{2}$, providing it be lefs than c, (as it may be) ac will be greater than aa. Again; Whether we take $a - \frac{c}{2}$ or $\frac{c}{2} - a$, the Square of it is $a^2 - ac + \frac{cc}{4} = \frac{cc}{4} - ac + a^2$, which is alfo $= \frac{cc}{4} - \overline{ac - aa} = \frac{cc}{4} - R$, (becaufe $ac - aa = R$.) But the Root being real or pofitive, fo muft the Square be; *i. e.* $\frac{cc}{4}$ is greater than R. Or, if $a = \frac{c}{2}$, then $a^2 = \frac{cc}{4}$, and $2 a = c$; alfo $2 aa = ac$; confequently $ac - aa = 2aa - aa = aa$; and $\frac{cc}{4} = ac - aa = R$. So that R can never be greater than $\frac{cc}{4}$, tho' it may be either equal or lefs. And *obferve*, if they are equal, then there is but one Solution, *viz.* $a = \frac{c}{2}$; for here both the Solutions coincide.

For the fecond thing, *viz.* that $\frac{c}{2}$ is greater than $\overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$; confider that $\frac{cc}{4}$ is greater than $\frac{cc}{4} - R$, and confequently $\frac{c}{2}$ (the Square Root of $\frac{cc}{4}$) is greater than $\overline{\frac{cc}{4} - R}\,|^{\frac{1}{2}}$. Or, we have this in the very Suppofition; for, by the firft Part of the Demonftration of this Rule, the
<div align="right">Value</div>

Value of a is $= \frac{c}{2} - \overline{\frac{cc}{4} - R}\Big|^{\frac{1}{2}}$, upon that very Suppofition that $\frac{c}{2}$ is greater than a; whence it was fhewn that $\overline{\frac{cc}{4} - R}\Big|^{\frac{1}{2}} = \frac{c}{2} - a$, and confequently $\frac{c}{2}$ greater than $\overline{\frac{cc}{4} - R}\Big|^{\frac{1}{2}}$.

S C H O L I U M S.

1. If the Difference of the Square and Multiple of the Root is given, without determining which of them is greateft, then we muft try both Rules.

2. This Problem is what the *Algebraifts* call, *Extracting the Root of an adfected Square*, (*i. e.* wherein the Number given is the Sum or Difference of a Square, and a certain Multiple of the Root; whofe Multiplier is alfo given) The Solutions explained are all that are real and pofitive; yet the Algebraick Art confiders two Roots or Solutions in every Cafe: But the Roots that I have not explained are only negative and imaginary; and to fay any thing farther about them, were to exceed the Limits prefcribed to this Work; and for the fame Reafon I am obliged to fpeak nothing of extracting the Roots of higher Powers that are adfected.

C H A P. III.

The Arithmetick *of* S u r d s.

W H A T a *Surd* is has been already explained : It has been demonftrated that every Number has not a perfect and determinate Root; but yet that we can find an Approximate Root within any affignable Difference of a true and compleat one; fo that it may be truly faid, that the Quantity which hinders any Number from being a compleat Power of any kind, is infinitely little; or that a Quantity infinitely little (or lefs than any affigned one) being taken from the Quantity expreffed by any given Number, the Remainder is a Quantity expreffible by a Number (of the fame Parts) which is a true Power of the Order propofed ; with this Difference, that it will be a fractional Power and not an integral. Now fince *Surds*, or indeterminate Roots, can be determined infinitely near; and fince the indeterminate Series goes on by a certain Law or Condition, it may be conceived as fome whole and compleat thing of its own kind; and

therefore, taking Surds under the general Expreffion of Roots, as $N^{\frac{1}{m}}$, we may apply all the *Theory* of Chap. I. and all the *Operations* of Arithmetick to them, as if they were determinate : For thus we can form general Ideas of Sums, Differences, Products and Quotes of Surds, imagined under the Notion of compleat Quantities of their own kind, the fame way as we do of rational or determinate Roots expreffed after the fame general manner. And hereby we can difcover certain Connections and Relations of Quantities thus reprefented, which may lead us to fome other particular Truths we would difcover.

It's true indeed, that as to any actual Operation with fuch Roots it can only be made in an imperfect manner, by way of Approximation; yet fince we can approximate or determine the Root fo far, that taking it for true and compleat, the Error it can make in any Operation or Conclufion fhall be within any affignable Difference of what it would be if the compleat Value of the Surd could poffibly be determined and ufed in the Operation : Therefore our arguing with them as we do with rational and determinate Roots, is fo far at leaft juft and conclufive; and is indeed abfolutely fo, taking them in general and abftractly.

To illuftrate this by a few *Examples*: The Sum of $\sqrt{8}$ and $\sqrt{12}$ may be expreffed in general $\sqrt{8} + \sqrt{12}$, whatever thefe are in themfelves; and if we would apply this by an actual Operation, then we can approximate each of thefe Roots fo near, that their Sum fhall want lefs than any affigned Difference of what it would be if the Roots could be determined.

Example 2. To multiply $\sqrt{8}$ by 3, it may be expreffed thus, $3 \times \sqrt{8}$; and by Approximation we can find a determinate Number for $\sqrt{8}$, which multiplied by 3, the Product fhall want lefs than any affigned Difference of what it would be if the Root could be compleatly determined.

Example 3. The Product of $\sqrt{8}$ and $\sqrt{6}$ may be expreffed $\sqrt{8} \times \sqrt{6}$; and by Approximation we can take $\sqrt{8}$ and $\sqrt{6}$ fo near, that multiplying them together at every Step, the Products fhall ftill increafe and come within any affigned Difference of what it would be were the Roots determined.

Again: Tho' Surds can never be reduced to determinate Numbers (for then they were not Surds), yet in many Cafes their Sums, Differences, Products and Quotes can be expreffed after different ways (by means of the *Theory* explained in *Chap.* I.), which are more or lefs fimple and convenient; fo that what by the more general Rules can be expreffed only by Signs of Addition, &c. may be expreffed more fimply, either by one Surd, or by an Expreffion partly furd, partly rational, and in fome Cafes altogether rational. Now to this tends the more particular Practice or Arithmetick of Surds; which depending upon certain different Forms in which the fame Surd may be expreffed, therefore the firft thing to be explained is *The Reduction of Surds*; the Demonftration of which depends upon the *Theorems* in *Chap.* I. applied to *Surds*.

Obferve alfo, That all the following Practice is equally applicable to rational Roots expreffed in the general radical Form; for when we take general Expreffions they comprehend all poffible Cafes; and the Practice propofed is often as convenient with refpect to Rationals as Surds, becaufe it's convenient fometimes to exprefs even rational Numbers in this radical Form; and therefore, tho' it's commonly called the *Arithmetick of Surds*, it were as proper to call it the *Arithmetick of Radicals*.

Reduction of Surds (*or* Radicals).

C A S E I. To exprefs any Number in Form of a Surd (*i. e.* in a radical Form). R U L E: Raife the given Number to the Power of the Surd, and then apply the Surd Index, thus; $8 = 64^{\frac{1}{2}}$, for $8 \times 8 = 64$. Univerfally, $A = \overline{A^n}|^{\frac{1}{n}}$.

The *Reafon* is manifeft from the *Definitions*, and *Ax*. I.

C A S E II. To reduce a Surd with a Mixt Index (*i. e.* whofe Numerator is greater than 1) to another, having a fimple radical Index (*i. e.* whofe Numerator is 1.) R U L E: Involve the Number given to a Power whofe Index is the Numerator of the Mixt Index, and to the Number found apply the Denominator radically. *Example*: $8^{\frac{2}{3}} = 64^{\frac{1}{3}}$; for $8^2 = 64$. Univerfally, $A^{\frac{n}{r}} = \overline{A^n}|^{\frac{1}{r}}$.

The *Demonftration* of this is plainly in the Definition: For $A^{\frac{n}{r}}$ expreffes the r Root of the n Power (which is alfo the n Power of the r Root by *Theor.* X.)

C A S E III. To reduce two Unlike Surds to Like: *i. e.* having two unlike Surds of the fame or different Numbers, to find other two Surds equal refpectively to the given ones, but having the fame Index, and that alfo the leaft poffible; and fuch too, that the Numbers under the common Index be the leaft poffible.   R U L E.

R U L E. Reduce their Indexes to one common Denominator by the Rules of Fractions. Again, find the greatest common Measure to both the new Numerators, (*i. e.* the greatest Number which will divide them both without a Remainder,) by the Method taught in the Reduction of the Numerator and Denominator of a Fraction to their least Terms; make that common Measure the common Numerator to the common Denominator before found: The lowest Terms of this Fraction is the common Index sought.

Again; Divide the new Numerators mentioned by their greatest common Measure, and mark the Quotes; then involve each given Number to a Power whose Index is the respective Quote; and that is the Number to which if the common Index is applied the Case is compleatly solved.

Example: To reduce $8^{\frac{1}{2}}$ and $15^{\frac{1}{3}}$ to Like Surds, with the other Conditions proposed.

1. They are $8^{\frac{3}{6}}$ and $15^{\frac{2}{6}}$, by reducing the Indexes $\frac{1}{2}$ and $\frac{1}{3}$ to one Denominator; Then the greatest common Measure of the Numerators 3, 2, is 1, and the common Index is $\frac{1}{6}$; and to have Numbers to which it must be applied, I raise 8 to the 3d Power, and 15 to the 2d, (for here the common Measure of 3 and 2 is 1, which makes the Quotes the same,) these Powers are 512, 225; wherefore the Surds sought are $512^{\frac{1}{6}} = 8^{\frac{1}{2}}$, and $225^{\frac{1}{6}} = 15^{\frac{1}{3}}$.

Example 2. To reduce $4^{\frac{2}{3}}$ and $5^{\frac{4}{7}}$: They are first $4^{\frac{14}{21}}$, $5^{\frac{12}{21}}$, then the greatest common Measure of 14, 12, is 2; and so $\frac{2}{21}$ is the common Index, which is in its least Terms. Again; The Numerators 14, 12, divided by their greatest Measure 2, the Quotes are 7, 6, and $4^7 = 16384$. $5^6 = 15625$. Then lastly, $16384^{\frac{2}{21}} = 4^{\frac{2}{3}}$. $15625^{\frac{2}{21}} = 5^{\frac{4}{7}}$.

Example 3. To reduce $3^{\frac{1}{6}}$ and $4^{\frac{1}{3}}$: They are first $3^{\frac{3}{18}}$, $4^{\frac{6}{18}}$, and the greatest Measure of the Numerators 3, 6, being 3, the common Index is $\frac{3}{18} = \frac{1}{6}$ in its least Terms; then 3, 6, divided by 3, the Quotes are 1, 2; and $3^1 = 3$: $4^2 = 16$: Wherefore, lastly, $3^{\frac{1}{6}}$, $16^{\frac{1}{6}}$ are the Surds sought.

D E M O N. Let $A^{\frac{r}{s}}$, $B^{\frac{n}{u}}$ be any two Surds, (where, if *r* or *n* are 1, the Surds are simple.) These are first equal to $A^{\frac{ru}{su}}$, $B^{\frac{ns}{su}}$ by Reduction of the Indexes to one Denominator, (*Theor.* XI. *Ch.* I.) Suppose *m* to be the common Measure to *ru*, *ns*, and let the Quotes be $ru \div m = x$; $ns \div m = y$; so that $ru = mx$, and $ns = my$: Then the Surds are $A^{\frac{mx}{su}}$, $B^{\frac{my}{su}}$, that is, (by *Theor.* XII.) $\overline{A^x}^{\frac{m}{su}}$, $\overline{B^y}^{\frac{m}{su}}$; which is exactly the Expression of the Rule, supposing *m* the greatest common Measure of *ru*, *ns*, and $\frac{m}{su}$ to be in its least Terms; or if it's not in lowest Terms, yet its lowest Terms being put in its Place makes an equivalent Expression (*Theor.* XI.) Observe also, that tho' *m* is not the greatest common Measure of the new Numerators *ru*, *ns*, yet we have the Surds reduc'd to Like Surds, tho' not in the lowest Expressions; which it's plain will then, and then only happen when *m* is the greatest common Measure.

S C H O L I U M S.

1. It will be the same thing if we first reduce the given Surds to simple Indexes, if they are Mixt, and then reduce these new Indexes to one Denominator, and go on with the rest as in the Rule.

I   2. When

2. When there are more Surds propofed, the Operation and the Reafon of it is the fame; except that we have not yet learn'd how to find the greateft common Meafure to 3 or more Numbers (which you'll find in *Book* IV. *Ch.* I.) and therefore, till that be learn'd, we muft be content to reduce the Surds to Likes, tho' not in their loweft Terms, by ufing 1 as a common Meafure, which makes the Dividend and Quote the fame.

C A S E IV. To reduce a Surd having a fimple Index to lower Terms; *i. e.* to an equivalent Expreffion in which there is a fimilar Surd of a leffer Number multiplied into fome rational Number.

R U L E. Among the Numbers greater than 1, which meafure the given Number, (or Surd Power) feek one which is a fimilar and rational Power, by which divide the given Number: Take the Quote, and to it apply the given Index; and multiply that Root by the Root of the Divifor: This Product is the Expreffion fought.

Example 1. $8^{\frac{1}{2}} = 2 \times 2^{\frac{1}{2}}$; for $8 \div 4 = 2$, and $4^{\frac{1}{2}} = 2$.

Example 2. $648^{\frac{1}{3}} = 2 \times 81^{\frac{1}{3}} = 3 \times 24^{\frac{1}{3}}$; for $648 \div 8 = 81$, and $8^{\frac{1}{3}} = 2$; whence, by the Rule, the 1ft Solution is $2 \times 81^{\frac{1}{3}}$. Again; $648 \div 27 = 24$, and $27^{\frac{1}{3}} = 3$; whence the 2d Solution is $3 \times 24^{\frac{1}{3}}$.

D E M O N. Suppofe $A \div D^n = B$, fo that $A = B \times D^n$, then is $A^{\frac{1}{n}} = \overline{B \times D^n}^{\frac{1}{n}}$, *(Ax.* I.) and $\overline{B \times D^n}^{\frac{1}{n}} = D \times B^{\frac{1}{n}}$, (*Theor.* III.) which is precifely conform to the Rule; A reprefenting the given Number.

S C H O L I U M S.

1. If the Power by which we meafure the given Number is the greateft Like Power which meafures it, then we find the loweft Terms of the given Surd.

2. As to the finding the Numbers that meafure any given Number, you'll have it more particularly explained in *Book* IV. *Chap.* I. Here we fuppofe thefe to be given; becaufe from the Nature of the Thing this Rule for finding them is obvious, *viz.* To try all the Numbers not exceeding the half of the given Number; for all of thefe which meafure it, together alfo with the Quotes, make all the Numbers that meafure it. But unlefs thefe Meafures that ferve the prefent Problem are obvious, the finding them out is more Trouble than is always neceffary.

3. If the given Surd has a mixt Index, the fame kind of Reduction may be performed by reducing it firft to a Surd with a fimple Index; and then applying the prefent Rule. And again; If the Number under the radical Sign in the Anfwer thus found, is a rational Power of the Order expreffed by the Numerator of the given Index, then by taking the Root of it we may alfo reduce the whole to a Surd with the given mixt Index. Thus, $192^{\frac{2}{3}}$ is firft $= 36864^{\frac{1}{3}}$ ($= 192^{\frac{2}{3}}$); which again, reduced is $= 16 \times 9^{\frac{1}{3}}$; for $16^3 = 4096$, and $36864 \div 4096 = 9$: And becaufe 9 is a Square, therefore $9^{\frac{1}{3}} = 3^{\frac{2}{3}}$; and hence $16 \times 9^{\frac{1}{3}} = 16 \times 3^{\frac{2}{3}} = 192^{\frac{2}{3}}$.

C O R O L L. Hence we fee plainly, That one fimilar Surd may be a Multiple or aliquot Part of another. But obferve that in applying this to Practice, all we can make of it is, That the greater Surd approximate to a certain degree, and divided by the other approximate to the fame degree, the Quote will be within a certain Difference of that Number, which, by this Reduction, appears to be the Quote: But being approximate nearer and nearer *in infinitum*, the Quote will be nearer *in infinitum* to that other, which we here call the True and Compleat Quote. But if the fame Dividend be divided by

any

any other Surd or Number whatever, the Quote can be brought to exceed that true Quote, or will never be brought within an affignable Difference of it; and therefore it's juftly called the true Quote of thefe two Surds.

C A S E V. To reduce any two Surds to Expreffions, having a common Surd; *i. e.* to Expreffions that are Products of rational Numbers into a common Surd.

R U L E. Reduce the given Surds to the fame fimple Index, if they are not fo already (by *Cafe* 3.): Then find the greateft common Meafure of the Powers (or Numbers under the radical Signs); and taking the Quotes, examine by Extraction if they are rational and fimilar Powers of the Order expreffed by the Denominator of the common fimple Index; if they are, their Roots are the rational Numbers fought; and the furd Root of the common Meafure is the furd Part fought: But if thefe Quotes are not fuch fimilar Powers, the Queftion is impoffible.

Example: To reduce $12^{\frac{1}{2}}$ and $27^{\frac{1}{2}}$: The greateft common Meafure of 12, 27, is 3, and the Quotes are 4, 9. which being rational Squares, I take their Roots 2, 3, and multiply them into the common Surd $3^{\frac{1}{2}}$, and the Expreffions fought are $2 \times 3^{\frac{1}{2}} = 12^{\frac{1}{2}}$, and $3 \times 3^{\frac{1}{2}} = 27^{\frac{1}{2}}$.

D E M O N S T R. Let $A^{\frac{1}{n}}$, $B^{\frac{1}{n}}$ be the given Surds (or the Expreffions to which they are reduced): Suppofe $A \div m = a^n$, and $B \div m = b^n$, fo that $A = m \times a^n$, and $B = m \times b^n$; then $A^{\frac{1}{n}} = \overline{m \times a^n}|^{\frac{1}{n}}$ (*Ax.* I.) $= a \times m^{\frac{1}{n}}$, (*Theor.* I. *Cor.*) Alfo $B^{\frac{1}{n}} = \overline{m \times b^n}|^{\frac{1}{n}}$, $= b \times m^{\frac{1}{n}}$, which is exactly according to the Rule, fuppofing *m* to be any common Meafure: And the Reafon why it's in the Rule called the greateft common Meafure, is, becaufe if the greateft will not quote fimilar rational Powers, none of the other common Meafures will; which remains to be demonftrated. Thus; take the given Surd Powers fractionwife, $\frac{A}{B}$; This is not an immediate fractional Power of the Order *n*, becaufe by Suppofition, neither A or B are rational Powers of that Order; but if any other Fraction equivalent to $\frac{A}{B}$ is an immediate Power of that Order, the leaft Terms of $\frac{A}{B}$ will be fo; and if the leaft Terms are not fo, no other Terms can be fo, (as has been demonftrated in the Rule for Extracting the Roots of Fractions) *i. e.* if A, B, being divided by their greateft common Meafure do not give for Quotes fimilar rational Powers of the Order *n*, neither can their Quotes by any other common Meafure do fo.

S C H O L I U M S.

1. The greateft common Meafures quoting fimilar rational Powers, is a certain Character of the Problems being poffible, tho' none of the other common Meafures fhould make fuch Quotes; but if any of thefe others do fo, thefe would make fo many different Solutions to the Problem; in which this Difference is to be *obferved*, that the leffer the common Meafure is which we ufe, the leffer Terms will the Solution be in, as to the Surd Part: And the Reafon why we chufe the greateft Meafure in the Rule is, becaufe that tho' from any other Meafure's giving Quotes which are rational Powers we are fure that the Problem is poffible, yet we can conclude it impoffible from no other but the greateft common Meafure giving Quotes which are not Like Powers.

2. If the two Surd Powers are Fractions, then reduce them to any common Denominator, and if the new Numerators are reducible according to this Rule, fo are the given
Surds.

Surds. One Example will fhew this: Suppofe $\overline{\frac{12}{15}}^{\frac{1}{2}}$, $\overline{\frac{3}{15}}^{\frac{1}{2}}$, the Numerators 12, 3, taken

radically, *viz.* $12^{\frac{1}{2}}$, $3^{\frac{1}{2}}$, are reducible to thefe, $2 \times 3^{\frac{1}{2}}$, $1 \times 3^{\frac{1}{2}}$, or $3^{\frac{1}{2}}$; wherefore the gi-

ven Surds are reduced to thefe, $2 \times \overline{\frac{3}{15}}^{\frac{1}{2}}$, and $\overline{\frac{3}{15}}^{\frac{1}{2}}$.

Again; If either the Numerators or Denominators of two Fractions, affected with a fimple radical Sign, or reduced to that State, are rational Powers of the Order expreffed by the Denominator of the Index. The Fractions need not be reduced to a common Denominator; for we need only examine if the other Terms are reducible to a common Surd Power: Thus; Suppofe $\overline{\frac{50}{16}}^{\frac{1}{2}}$, $\overline{\frac{72}{25}}^{\frac{1}{2}}$: Here the Denominators, 16, 25, are Squares, whofe Roots are 4, 5. Again; $50^{\frac{1}{2}} = \overline{2 \times 25}^{\frac{1}{2}} = 5 \times 2^{\frac{1}{2}}$. Alfo $72^{\frac{1}{2}} = \overline{2 \times 36}^{\frac{1}{2}} = 6 \times 2^{\frac{1}{2}}$, whence it's plain that the given Surds are $2^{\frac{1}{2}} \times \frac{5}{4}$, $2^{\frac{1}{2}} \times \frac{6}{5}$. And had the given Surds been $\overline{\frac{16}{50}}^{\frac{1}{2}}$, $\overline{\frac{25}{72}}^{\frac{1}{2}}$, the Solution is $\frac{4}{5} \times \overline{\frac{1}{2}}^{\frac{1}{2}}$, $\frac{5}{6} \times \overline{\frac{1}{2}}^{\frac{1}{2}}$; for $16^{\frac{1}{2}} = 4$, and $50^{\frac{1}{2}} = 5 \times 2^{\frac{1}{2}}$, then is $\overline{\frac{16}{50}}^{\frac{1}{2}} = \frac{4}{5 \times 2^{\frac{1}{2}}} = \frac{4}{5} \times \frac{1}{2^{\frac{1}{2}}}$, or $\frac{4}{5} \times \overline{\frac{1}{2}}^{\frac{1}{2}}$, and fo of the other.

3. This Cafe is commonly called *Finding, if two Surds are commenfurable*; *i. e.* if they have a common Meafure, or if there is any Surd which meafures or is an aliquot Part of each of them; whereby they are reducible to Expreffions which are the Products of that common Surd into the refpective Quotes. *Obferve* alfo, That the Meafure of a Surd muft be a Surd, which is manifeft; for if any rational Number fhould meafure a Surd, or be an aliquot Part of it, then that aliquot Part and its Denominator (or the Meafure and Quote) would produce the Dividend, *i. e.* two rational Numbers would produce a Surd, which is impoffible.

The Ufe of thefe Reductions in the common Operations of Addition, *&c.* I *fhall briefly fhew thus:*

In *Addition and Subtraction of* Surds.

If one Surd is to be added to or fubtracted from another, and if they are commenfurable, *i. e.* reducible to a common Surd, by *Cafe* 5. this Reduction being made, or if the given Expreffions are of this kind, the Sum or Difference of the rational Parts multiplied into the common Surd Part is the Sum or Difference fought, in a more fimple and convenient Form than connecting the given Numbers by the general Signs of Addition and Subtraction, which is the general Rule for all other Cafes.

Example 1. $8^{\frac{1}{2}} + 50^{\frac{1}{2}} = 2 \times 2^{\frac{1}{2}} + 5 \times 2^{\frac{1}{2}} = 7 \times 2^{\frac{1}{2}}$.

Example 2. $54^{\frac{1}{3}} - 16^{\frac{1}{3}} = 3 \times 2^{\frac{1}{3}} - 2 \times 2^{\frac{1}{3}} = \overline{3 - 2} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3}}$.

The Sum or Difference of two Square Roots may be alfo exprefs'd. *Thus:* Take the Surd Powers, or Numbers under the radical Sign; to the Square Root of double their Product, add their Sum, or fubtract that Root from this Sum; the Square Root of this Sum or Difference expreffes the Sum or Difference fought. *Example:* $5^{\frac{1}{2}} + 3^{\frac{1}{2}} = \overline{8 + 30^{\frac{1}{2}}}^{\frac{1}{2}}$, and $5^{\frac{1}{2}} - 3^{\frac{1}{2}} = \overline{8 - 30^{\frac{1}{2}}}^{\frac{1}{2}}$.

DEMON,

DEMON. Suppofe $A^{\frac{1}{2}} = a$, $B^{\frac{1}{2}} = b$; then is $ab = A^{\frac{1}{2}} \times B^{\frac{1}{2}} = \overline{AB}^{\frac{1}{2}}$ (*Theor.* 3.) and $A^{\frac{1}{2}} + B^{\frac{1}{2}} = a + b = \overline{a^2 + b^2 + 2ab}^{\frac{1}{2}}$. Alfo $A^{\frac{1}{2}} - b^{\frac{1}{2}} = a - b = \overline{a^2 + b^2 - 2ab}^{\frac{1}{2}}$, which is exactly according to the Rule.

For Multiplication and Divifion of S U R D S.

If one Surd is to be multiplied or divided by another; then if they are unlike, reduce them to Likes, and examine if they are commenfurable, *i. e.* reducible to Expreffions, wherein the fame common Surd is multiplied into rational Numbers; and if it is fo, multiply or divide the rational Parts, the Product multiplied again into the Square of the common Surd is the Product fought: fo that if the given Surds are Square Roots, the Product is rational. But in Divifion the Quote of the rational Parts alone is the Quote fought; which is therefore rational.

Example. $72^{\frac{1}{2}} \times 32^{\frac{1}{2}} = 48$. For $72 = 8 \times 9$, and $9^{\frac{1}{2}} = 3$; therefore $72^{\frac{1}{2}} = \overline{8 \times 9}^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 3 \times 8^{\frac{1}{2}}$. Again, $32^{\frac{1}{2}} = \overline{4 \times 8}^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 2 \times 8^{\frac{1}{2}}$: fo that $72^{\frac{1}{2}} \times 32^{\frac{1}{2}} = 3 \times 8^{\frac{1}{2}} \times 2 \times 8^{\frac{1}{2}} = 3 \times 2 \times 8^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 6 \times 8 = 48$. And $72^{\frac{1}{2}} \div 32^{\frac{1}{2}} = 3 \times 8^{\frac{1}{2}} \div 2 \times 8^{\frac{1}{2}} = 3 \div 2$.

DEMON. Suppofe $A^{\frac{1}{n}} = a \times R^{\frac{1}{n}}$, and $B^{\frac{1}{n}} = b \times R^{\frac{1}{n}}$; then $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = ab \times R^{\frac{1}{n}} \times R^{\frac{1}{n}} = ab \times R^{\frac{2}{n}}$. Wherefore if $n = 2$, the Product is abR. Alfo $A^{\frac{1}{n}} \div B^{\frac{1}{n}} = a \times R^{\frac{1}{n}} \div b \times R^{\frac{1}{n}} = a \div b$.

S C H O L I U M S.

1. To multiply fimilar Surds: If we multiply the Surd Powers, and apply the fame Index to that Product; this expreffes the Product fought more fimply, than by the general Sign of Multiplication. Thus: $A^{\frac{1}{n}} \times B^{\frac{1}{n}} = \overline{AB}^{\frac{1}{n}}$ (*Theor.* 3.) Again; if this is reducible, bring it to loweft Terms, and you'll have in many Cafes the fame Product that the preceding Rule brings out; and it's always the beft we can make of it, when the given Surds are not commenfurable. In the preceding Example, $72^{\frac{1}{2}} \times 32^{\frac{1}{2}} = \overline{72 \times 32}^{\frac{1}{2}} = 2304^{\frac{1}{2}} = 48$.

2. If a rational Number is multiplied into a Surd, it may be fometimes convenient to exprefs it altogether radically; for which you have a Rule in *Theo.* 3. *Cor.* Thus: Raife the rational Part to the Power whofe Index is the Denominator of the furd Part, and multiply this Power into the furd Power; then apply the radical Index. *Exam.* $A \times B^{\frac{1}{n}} = \overline{A^n B}^{\frac{1}{n}}$, and $A \times B^{\frac{1}{n}} = \overline{A^n Br}^{\frac{1}{n}}$, for $B^{\frac{1}{n}} = Br^{\frac{1}{n}}$.

3. If a Surd and rational Number are multiplied; and if the Surd is reducible to lower Terms, the whole Product is fo. Thus: $6 \times 45^{\frac{1}{2}} = 6 \times 3 \times 5^{\frac{1}{2}} = 18 \times 5^{\frac{1}{2}}$; for $45 = 9 \times 5$, and $45^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 3 \times 5^{\frac{1}{2}}$.

You may apply the fame Obfervations to Divifion. So for the 1ft, $72^{\frac{1}{2}} \div 32^{\frac{1}{2}} = \overline{72 \div 32}^{\frac{1}{2}} = \overline{9 \div 4}^{\frac{1}{2}}$, or $\overline{\frac{9}{4}}^{\frac{1}{2}} = \frac{3}{2}$.

And from this Example, wherein the Product or Quote becomes rational, we have a farther remarkable Proof of the Reafonablenefs and Ufefulnefs of our treating Surds, and working with them in all refpects as with Rationals or compleat Roots; for if any other Number than 48 is fuppofed to be the Product of $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$, we can prove it to be falfe.

D d   Thus:

Thus: Since $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ can be Approximate *in infinitum*, the Products of thefe approximate Roots will grow *in infinitum* towards a certain Limit; which muft neceffarily be $\overline{72 \times 32}^{\frac{1}{2}} = 48$; becaufe if $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ are both rational, then is $\overline{72 \times 32}^{\frac{1}{2}}$ rational, being equal to $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$ (*Theor.* 3.) And tho' $72^{\frac{1}{2}}$, $32^{\frac{1}{2}}$ are furd, yet being infinitely approximable, their Product will grow infinitely near to $\overline{72 \times 32}^{\frac{1}{2}} = 48$; which is therefore the true Limit or compleat Value of $72^{\frac{1}{2}} \times 32^{\frac{1}{2}}$, beyond which it cannot poffibly grow; nor can it be fuppofed lefs, becaufe we can approximate $72^{\frac{1}{2}}$ and $32^{\frac{1}{2}}$ fo far, that the Product fhall exceed any affignable Number lefs than $48 = \overline{72 \times 32}^{\frac{1}{2}}$; for elfe they were not infinitely approximable, as is fuppofed and demonftrated.

CHAP. IV.

Containing feveral THEOREMS *relating to the* Powers *of Numbers.*

IN the following *Theorems* and *Corollaries* are comprehended all the Propofitions of the *Second Book* of EUCLID that are applicable to Numbers, excepting four, which are in effect already explained in this Work; but that you may fee them all in this Place, I briefly repeat thefe four.

1. If one Number A, (or Line, as it is in *Euclid, Book* II. *Theor.* 1.) is multiplied feverally into all the Parts of another $B = a + b + c$, &c. the Sum of the Products is the Product of the two Wholes; thus $A a + A b + A c$, &c. $= A B$. This you have already in *Lemma* 3 *Ch* 5. *Book* I. which, obferve, is equally applicable to Fractions and Integers.

2. If any Number is multiplied into all its own Parts feverally, the Sum of the Products is equal to the fquare of the Whole; which is the Confequence of the laft; thus, if $N = a + b$, then is $N^2 = N a + N b$ (*Euclid*, Theor. 2.)

3. If a Number is divided into two Parts, the Product of the Whole and one Part, is equal to the Sum of the Square of this Part, and the Product of the Parts: This is alfo a Confequence, or particular Cafe of the 1ft: Thus, if $N = a + b$, then $N a = a^2 + a b$, (*Euclid*, Theor. 3.)

4. If a Number is divided into two Parts, the Square of the Whole is equal to the Sum of the Squares of the Parts, and twice the Product of the Parts: This proceeds alfo from the 1ft: Thus, if $N = a + b$, then $N^2 = a^2 + b^2 + 2 a b$, (*Euclid*, Theor. 4.)

The reft of *Euclid* you have in the following *Theorems*.

THEOREM I.

THE Square of any Number is equal to the Difference of the Products of that Number, multiplied into any greater Number, and into the Difference of thefe Numbers. Or it is equal to the Sum of the Products of that Number multiplied into any leffer, and into the Difference of thefe Numbers.

DEMONSTR. (1) Let two Numbers be a, $a + d$, wherein d is the Difference; then $a \times \overline{a + d} = a^2 + a d$; hence $a^2 = a \times \overline{a + d} - a d$. (2.) Let the Numbers be $a - d$, a, wherein d is alfo the Difference; then $a \times \overline{a - d} = a^2 - a d$, and $a^2 = a \times \overline{a - d} + a d$.

THEO:

THEOREM II.

THE Sum of the Squares of two Numbers is equal to the Sum of twice their Product and the Square of their Difference.

DEMONSTR. Let the Roots be a, b, then $a^2 + b^2 = 2ab + \overline{a-b}^2$; for $\overline{a-b}^2 = a^2 - 2ab + b^2$; whence the *Theorem* is manifest.

Or let the Roots be a, $a+b$, wherein b is the Difference; then is $a^2 + \overline{a+b}^2 = a^2 + a^2 + 2ab + b^2 = 2a^2 + 2ab + b^2 = 2a \times \overline{a+b} + b^2$.

SCHOLIUM. In this last Form we see plainly comprehended *Euclid's Theor.* 7. which is this; If a Number confist of two Parts, (a, b,) the Sum of the Squares of the Whole and of one Part, (*viz.* $\overline{a+b}^2 + a^2$) is equal to double the Product of the whole into this Part, and the Square of the other Part (*viz.* $2a \times \overline{a+b} + b^2$).

THEOREM III.

THE Sum of the Squares of two Numbers is equal to the $\frac{1}{2}$ Sum of the Squares of their Sum and Difference.

DEMONSTR. The two Numbers being a, b, then $a^2 + b^2 = \dfrac{\overline{a+b}^2 + \overline{a-b}^2}{2}$, for $\overline{a+b}^2 = a^2 + 2ab + b^2$, and $\overline{a-b}^2 = a^2 - 2ab + b^2$, and the Sum of these two Squares is $2a^2 + 2b^2$, whose $\frac{1}{2}$ is $a^2 + b^2$.

COROLL. The Sum of two Squares is double the Sum of the Squares of their half Sum and half Difference; for $a+b$, $a-b$, may represent any two Numbers, whose half Sum is a, and their half Difference is b; but we see above that $2a^2 + 2b^2 = \overline{a+b}^2 + \overline{a-b}^2$.

SCHOLIUMS.

1. This *Corrollary* is in effect the same as *Euclid, Theor.* 9. *viz.* If a Number is divided into two equal Parts, a, a, and into two unequal Parts $a+b$, $a-b$, (whose Sum is $2a$) the Sum of the Squares of the unequal Parts, (*viz.* $\overline{a+b}^2 + \overline{a-b}^2$) is equal to twice the Square of the half, (*viz.* $2 \times a^2$) and twice the Square of the middle Part, or half Difference of the unequal Parts (*viz.* $2 \times b^2$).

2. If we expres the supposed Numbers thus, a, $a+b$, then the *Theorem* is $a^2 + \overline{a+b}^2 = \dfrac{\overline{2a+b}^2 + b^2}{2}$. Also by taking b and $\overline{2a+b}$ for two Numbers, whose Sum is $2a + 2b$, and their half Sum $a+b$, and half Difference a, the preceding *Corollary* is thus expressed, $\overline{2a+b}^2 + b^2 = 2 \times \overline{a+b}^2 + 2 \times a^2$: Which is, in effect the same as *Euclid, Theor.* 10. *viz.* If a Number is equally divided into two Parts, a, a, and to the whole $2a$ another Number be added as b, the Square of the Sum, *viz.* $\overline{2a+b}^2$, and the Square of

the

the Number added, *viz.* b^2, are together equal to double the Squares of $\frac{1}{2}$ the 1ft Number, and of the Sum of that half and the Number added, *viz.* $2a^2 + 2 \times \overline{a+b}^2$.

THEOREM IV.

THE Sum of the Squares of two Numbers is equal to the Difference betwixt the Square of their Sum and double their Product.

DEMONSTR. The two Numbers being *a*, *b*, the *Theorem* is $a^2 + b^2 = \overline{a+b}^2 - 2ab$, for $\overline{a+b}^2 = a^2 + 2ab + b^2$, from which take $2ab$, remains $a^2 + b^2$.

Or thus; Let the two Numbers be $2a$, *b*, then $4a^2 + b^2 = \overline{2a+b}^2 - 4ab$.

COROLL. The Square of the Sum of two Numbers is 4 times their Product, more the Square of their Difference. This appears by making *a*, $a+b$, the two Numbers, whereby $2a+b$ is their Sum, and then adding $4ab$ to both Sides, whereby $\overline{2a+b}^2 = 4a^2 + 4ab + b^2 = 4 \times a \times \overline{a+b} + b^2$.

SCHOLIUM. This *Coroll.* is the fame in effect as *Euclid*, Theor. 8. *viz.* If a Number is divided into two Parts, as *a*, $a+b$; then 4 times the Product of the whole, and one Part, more the Square of the other Part, is equal to the Squares of the Sum of the whole and that Part.

THEOREM V.

THE Square of the Sum of two Numbers is equal to the Sum of the Square of one of them; and the Product of the other into the Sum of this other and double the former.

Alfo the Square of the Difference of two Numbers is equal to the Difference of the Square of one of them, and the Product of the other into the Difference of this other, and double the former.

DEMONSTR. 1. $\overline{a+b}^2 = a^2 + 2ab + b^2 = a^2 + \overline{2a+b} \times b$.

2. $\overline{a-b}^2 = a^2 - 2ab + b^2 = a^2 - \overline{2a-b} \times b$.

SCHOLIUM. The firft Part of this comprehends *Euclid*, *Theor.* 6. *viz.* If a Number is divided into two equal Parts *a*, *a*, and to the whole $2a$, another Number *b* is added; the Product of the Sum $2a+b$ by the Number added *b*, *viz.* $\overline{2a+b} \times b$, *&c.* together with the Square of $\frac{1}{2}$ the firft Number, *viz.* a^2, is equal to the Square of the Sum of this half Number, and the Number added, *viz.* $\overline{a+b}^2$

THEOREM VI.

THE Difference of the Squares of two Numbers is equal to the Product of their Sum and Difference.

DEMON. $\overline{a+b} \times \overline{a-b} = a^2 + ab - ab - b^2 = a^2 - b^2$.

COROLLARIES.

1. Of two unequal Numbers, $a+b$, $a-b$, the Square of half their Sum, *viz.* a^2 (for $2a$ is the Sum) is the Sum of their Product, *viz.* $a^2 - b^2$, and the Square of their Difference, *viz.* b^2.

SCHOL.

SCHOLIUM. This is *Euclid's Theor.* 5. *viz.* If a Number is divided into two equal Parts, a, a; and into two unequal Parts, $a+b$, $a-b$; the Product of the unequal Parts (*viz.* $\overline{a+b} \times \overline{a-b} = a^2 - b^2$) together with the Square of the middle Part, (*i. e.* of half their Difference, b^2) is equal to the Square of half the given Number, *viz.* a^2.

2. The Sum and Difference of two Numbers are the reciprocal aliquot Parts of the Difference of their Squares.

3. The Square of any Number may be produced after a new Method. Thus: Let the given Root be N, assume any other Number A; the Product of their Sum and Difference, which call p, is the Difference of their Squares, *i. e.* $p = N^2 - A^2$, or $A^2 - N^2$. Hence $N^2 = A^2 + p$, or $A^2 - p$.

4. Take any Number A, and make a Series from it continually decreasing by 1, till the least Term be 1; also a Series increasing by 1, to the same Number of Terms; multiply the corresponding Terms of the two Series together; the Products make a Series decreasing in such a manner, that comparing each Product to the Square of A, the Differences are the Series of Squares of the natural Progression, 1, 2, 3, &c. the Deduction of which from the *Theorem* is easy. Thus: $\overline{A-n} \times \overline{A+n} = A^2 - N^2$. But the Differences betwixt A,

$$A = 8.\ 7.\ 6.\ 5.\ 4.\ 3.\ 2.\ 1.$$
$$A = 8.\ 9.10.11.12.13.14.15.$$
Products - - - $64.63.60.55.48.39.28.15.$
Differs from 64. - - $1.\ 4.\ 9.16.25.36.49.$

and the several Terms of the Series, are, by Supposition, 1, 2, 3, 4, &c. *that is,* n is gradually 1, 2, 3, &c. Consequently the Differences of the several Products of the corresponding Terms of the two Series from A^2 the first Product, are gradually the Squares of these Roots, 1, 2, 3, &c. Hence again,

5. We have this Rule for summing the Series of the Squares of the natural Progression 1, 2, 3, &c. *viz.* Take any Number A greater than n, the greatest of the Roots whose Squares are to be summed; then beginning at $A-1$, and $A+1$, continue a Series downwards from $A-1$, and upwards from $A+1$, with the common Difference 1, till the Number of Terms be n; then taking the Products of the two Series as before, subtract their Sum from $n \times A^2$; the Remainder is the Sum sought. The Reason is plain; for the Sum of the Products is n times A^2, wanting the Sum of the Series of Squares 1, 4, 9, &c. taken to a Number of Terms equal to n; therefore also the

$$A-1 : A-2 : A-3 : A-4, \&c.$$
$$A+1 : A+2 : A+3 : A+4, \&c.$$
$$A^2-1 : A^2-4 : A^2-9 : A^2-16, \&c.$$

Sum of the Squares is $n \times A^2$ wanting the Sum of the Products.

THEOREM VII.

The Sum of any Number of different Powers of the same Root, which stand all next together in the Series or Order of Powers, [*i. e.* whose Indexes follow one another in the natural Series of Numbers 1, 2, 3, &c. but beginning at any Power, or Place of the Series] is equal to the Quote of the Difference of the least of these Powers, and that next above the greatest of them, divided by the Difference of the Root and 1. Thus:

Example 1. $a^4 + a^5 + a^6 = \dfrac{a^7 - a^4}{a - 1}$

2. $a + a^2 + a^3 + a^4 = \dfrac{a^5 - a}{a - 1}.$

$$\dfrac{a + a^2 + a^3 + a^4, \&c.\ a^n.}{a - 1.}$$
$$\overline{a^2 + a^3 + a^4 + a^5, \&c. + a^{n+1},}$$
$$\overline{a + a^2 + a^3 + a^4, \&c. + a^n.}$$
$$\overline{a^{n+1} - a.}$$

Ex. 3. $a + a^2 + a^3, \&c. + a^n = \dfrac{a^{n+1} - a}{a - 1}.$

4. $a^r + a^{r+1} + a^{r+2} +, \&c.\ a^{r+n} = \dfrac{a^{r+n+1} - a^r}{a - 1}.$

DEMON. Take the Series $a + a^2 + a^3 + a^4$ to a^n, multiply it by $a - 1$; the Product is $a^{n+1} - a$, as the annex'd Scheme of the Operation manifestly shews: For the given Series being multiplied by a, the Series of Products is the same as the given Series from the second Term, taking